ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
ENGINEERING AND TECHNOLOGY

DIRAC-LIKE HAMILTONIANS AND THE BERRY GAUGE FIELDS IN DIVERSE PHYSICAL SYSTEMS: FIELD THEORETICAL METHODS

Ph.D. THESIS

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DIRAC BENZERİ HAMİLTON YOĞUNLUKLARININ VE BERRY AYAR ALANLARININ ÇEŞİTLİ FİZİKSEL SİSTEMLERE UYGULAMALARI: ALAN KURAMI METODLARI

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For my dearest mother Birsen Gülsefa Elbistan

FOREWORD

First of all, i would like to thank my advisor, Prof. Ömer Faruk Dayı for his invaluable support during the preperation of the whole thesis. I also have to thank my family; to my wife Emine Günay Elbistan, my father Necati Elbistan and especially to my beloved mother Birsen Gülsefa Elbistan to whom i am indebted a life. This thesis and all are devoted to her.

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TABLE OF CONTENTS

Page

ABBREVIATIONS

QED	**:** Quantum Electrodynamics
TRB	**:** Time Reversal Breaking
TRI	**:** Time Reversal Invariant
CME	**:** Chiral Magnetic Effect
BF	**:** Background Field

LIST OF TABLES

Page

LIST OF FIGURES

Page

DIRAC-LIKE HAMILTONIANS AND THE BERRY GAUGE FIELDS IN DIVERSE PHYSICAL SYSTEMS: FIELD THEORETICAL METHODS

SUMMARY

In this study, the applications of Dirac-like Hamiltonians with and without mass terms and the related Berry gauge fields to diverse physical systems are discussed in terms of field theoretic methods.

In the first part, a pure gauge field is obtained through the diagonalization of the massive Dirac Hamiltonian. By projection onto the positive energy eigenstates, the Berry gauge field and the Berry field strength is defined. The related Chern numbers in terms of the Berry curvatures are defined in 2 and 4 space dimensions. Considering the $2+1$ dimensional Dirac particle interacting with the external electromagnetic field, the fermionic degrees of freedom are integrated out in the path integral formalism. At the first loop order, the effective action is found to be the topological Chern-Simons action. In the weak field limit, the coefficient of this action is shown to be the winding number of the free fermion propagator. This topological number is calculated by means of the projection operators. It is demonstrated that in both $2+1$ and $4+1$ dimensions these winding number are equal to the related Chern numbers.

In the second part, the Kane-Mele model in the presence of the Rashba spin-orbit interaction is considered. Kane and Mele, in their original article argued that the effect of the Rashba coupling would slightly modify the quantized value of the conductivity. Dealing with the Dirac spinors coupled to the external electromagnetic and spin gauge fields, the fermions are integrated through the path integral formalism. The resulting effective action is of the BF type. The coefficient of the effective action is the winding number of the fermion propagator which is modified with the S_z coupling. This coefficient gives the spin Hall conductivity. The calculation of the coefficient explicitly shows that the spin Hall conductivity altered slightly.

In the third part, the action of neutral Dirac particles coupled to electromagnetic field strength by the dipole interactions are proposed for the $3+1$ dimensional time reversal invariant topological insulators. By integrating out the fermionic fields, the θ vacuum action is obtained after a suitable normalization procedure as the effective action of the $3+1$ dimensional topological insulators. It is possible to obtain the BF type theories of the $3+1$ dimensional topological insulators within this scheme. For this purpose the auxilary gauge and antisymmetic tensor gauge fields are inserted into the action of the neutral particles and it is shown that after the path integral quantization the desired BF type action is acquired.

Beginning with the fourth part, the main focus will be on the applications of the massless Dirac Hamiltonian and the related Berry gauge fields. The $3+1$ dimensional Weyl Hamiltonian is diagonalized and a Berry gauge field is derived through the diagonalization procedure. It is shown that the Berry curvature results in the field of a monopole located at the center of the momentum space. The symplectic 2-form

for a positive energy Weyl particle interacting with the electromagnetic and Berry gauge fields is written. Phase space volume form is defined. The anomalous Liouville equation is obtained through the Lie derivative of the volume form. Using a proper phase-space distribution function, the non-conservation of the chiral current is shown within the chiral kinetic theory. By investigating the explicit form of the Liouvile equation, the equations of the motion for the phase space variables are observed. The chiral current is built and the chiral magnetic effect is derived. The same procedure is also explicitly applied to the $5+1$ dimensional case where the acquired Berry gauge fields are non-Abelian. The the whole formulation is generalized to all even $d+1$ dimensional spacetimes, hence the $d+1$ dimensional semiclassical chiral anomaly and chiral magnetic effect is acquired within the same formulation.

The last part is devoted to the topological concepts related to the Weyl Hamiltonian and the Berry gauge field. In $3+1$ and $5+1$ dimensions, the winding numbers for the free fermion propagators due to the Weyl Hamiltonians are defined. It is shown that the winding numbers are equal to the charge of the Dirac monopole located at the center of the momentum space. It is also shown that these winding number are the Chern numbers which are written in terms of the related Berry curvatures. The gauge field structure of the Dirac monopoles are explored. This procedure is generalized for all even $d+1$ dimensions.

DIRAC BENZERİ HAMİLTON YOĞUNLUKLARININ VE BERRY AYAR ALANLARININ ÇEŞİTLİ FİZİKSEL SİSTEMLERE UYGULAMALARI: ALAN KURAMI METODLARI

ÖZET

Bu çalışmada kütleli ve kütlesiz Dirac benzeri Hamilton yoğunlukları ve onlar vasıtasıyla türetilen Berry ayar alanlarının çeşitli sistemlere kuantum alan kuramları yöntemleri ile uygulamaları çalışılmıştır.

İlk yapılan çalışmalar, $2 + 1$ boyutlu zaman tersinmesi altında simetrik topolojik yalıtkanların etkin alan kuramları yoluyla fiziksel özelliklerinin tespit edilmesi ile ilgilidir. Tek atom kalınlığında, 2 boyutlu bir malzeme olan grafen, düşük enerji limitinde, momentum uzayında doğrusal bir enerji-momentum ilişkisine sahiptir. Bu ilişki nedeniyle grafendeki yük taşıyıcıları etkin olarak Dirac benzeri bir Hamilton yoğunluğu ile ifade edilir. Spin-yörünge etkileşiminin etkisiyle sözkonu Hamilton yoğunluğu kütle terimi de kazanır. Bu özellikler grafen gibi bir yoğun madde sisteminin, yüksek enerji fiziğinin argümanları ile incelenmesine olanak verir. Çalışmada Foldy-Wouthuysen dönüşümü ile kütleli Dirac Hamilton fonksiyonu köşegenleştirimiş ve ilgili köşegenleştirme matrisinin pozitif alt uzaya izdüşürülmesi ile Berry ayar alanları hesaplanmıştır. Berry ayar alanlarına ait eğrilikler elde edilmiş ve ilgili Chern sayıları tanımlanmıştır. Sürekli limitte, $2 + 1$ boyutlu, kütleli Dirac Lagrange yoğunluğu Feynman yol integrali metodu ile kuantize edilmiştir. Yol integralinde fermiyonik alanların integre edilmesiyle dış elektromanyetik alan cinsinden topolojik Chern-Simons etkin eylemi elde edilmiştir. Sözkonusu eylemin katsayısı topolojik bir değişmez olan fermiyon propagatörünün "dönme sayısı"dır. Serbest Dirac Hamilton yoğunluğunun propagatörü pozitif ve negatif alt uzaylara izdüşüren işlemciler vasıtasıyla ifade edilmiş ve bu sayede $2 + 1$ ve $4 + 1$ boyuttaki etkin eylemlerin katsayılarının, topolojik Chern sayılarına eşit olduğu gösterilmiştir.

Sonrasında, yine $2 + 1$ boyutlu Kane-Mele modeli bu sefer Rashba spin-yörünge etkileşiminin eklenmesi durumunda incelenmiştir. Rashba spin yörünge etkileşiminin mevcut olması spin operatörünün z yönündeki bileşeninin korunumlu bir büyüklük olmamasına yol açar. Yine de spin Hall evresi ve bu evrenin iletkenliğini benzer metotlarla incelemek mümkündür. $2 + 1$ boyutlu Kane-Mele modeline Rashba terimi de eklenmiş, oluşan Hamilton yoğunluğu köşegenleştirilmiştir. Köşegenleştirme matrisinden pozitif alt uzaya izdüşürülmek suretiyle Berry ayar alanı ve onu kullanarak Berry eğriliği tanımlanmıştır. $2+1$ boyutlu Dirac parçacığının elektromanyetik ve spin ayar alanları ile etkileşen kuramı ele alınmıştır. Spin akımının doğru tanımlanabilmesi için spin operatörünün z yönündeki bileşeni kuramda spin ayar alanının önüsıra yazılmış ve ilgili bölüşüm fonksiyonunda fermiyonlar integre edilmiştir. Bir ilmek mertebesindeki kuantum düzeltmeleri, dış elektromanyetik ve spin ayar alanları cinsiden topolojik kuramlar vermişlerdir. Bu kuramların katsayıları da ilgili fermiyon propagatörlerinin dönme sayılarına karşılık gelmektedirler. Bu katsayılar izdüşüm işlemcileri kullanılarak hesaplanmıştır. Zaman tersinmesi altında simetrik bir kuram ile ilgilenildiğinden sıfır olmayan tek katsayı zaman tersinme simetrisine sahip spin

Hall akımını veren eylemin katsayısıdır. Sözkonusu eylem BF tipi topolojik bir eylemdir. Bu eylemin elektromanyetik alana tepkisi spin Hall akımını vermektedir. Eylemin başındaki katsayının spin Hall olayının iletkenliğine eşit olduğu gösterilmiş ve analitik ve numerik yöntemlerin yardımı ile hesaplanmıştır. Sonuç, Kane-Mele tarafından tartışılan Rashba spin-yörünge etkileşiminin varlığının kuantize iletkenliği az da olsa bozması öngörüsü ile uyumludur.

Topolojik yalıtkanlarla ilgili son yapılan çalışma $3+1$ boyutlu zaman simetrisi altında değişmez topolojik yalıtkanları içermektedir. Sözkonusu yalıtkanların kuramı $4+1$ boyutlu kuramlardan boyut indirgeme yöntemi vasıtasıyla elde edilmektedir. Bu etkin kuramın parçacık fiziğinde de karşılaşılan θ vakum eylemi formatında olduğu bilinmektedir. Bu çalışmada ise doğrudan $3+1$ boyutta, yüksüz fermiyonlar ve onların çift kutup etkileşimlerini içeren bir Lagrange yoğunluğu önerilmiştir. Sözkonusu yüksüz sanki parçacıkların nasıl oluştuğu bilinmemektedir, orijinal Dirac parçacıkları olan elektronlar ve deşiklerin yüksüz bir fermiyon oluşturamayacağı da açıktır. Lakin böyle yüksüz fermiyonlar etkin olarak sanki parçacıklar şeklinde karşımıza çıkabilirler. Daha önceki çalışmalarda olduğu gibi yüksüz fermiyonların bölüşüm fonksiyonu içerisinde integre edilmeleri sonucu dış elektromanyetik alanlara bağlı topolojik eylem elde edilmiştir. $3+1$ boyutta çalışıldığı için parametrelerin uygun şekilde renormalize edilmeleri gerekmektedir. Elde edilen etkin eylem, beklenildiği üzere θ vakum eylemidir. Sözkonusu eylemin uzay-zamanda hangi katman üzerinde yazılacağı bilinmemekle birlikte, bazı tıkız katmanlar üzerinde uygun normalizasyon katsayısı ile birlikte, kuantize bir sayı vereceği bilinmektedir. Bu bağlamda $3+1$ boyuttaki kesirli topolojik yalıtkanlar da tartışılmıştır. Öte yandan $3+1$ boyutlu topolojik yalıtkanlar için BF tipi etkin eylemler de önerilmektedir. Sözkonusu etkin eylemleri yine yüksüz sanki parçacıkların kuramı vasıtasıyla elde etmenin mümkün olduğu da gösterilmiştir. Yöntem olarak yine ilgili yol integralindeki fermiyonlar integre edilerek bir halka mertebesindeki terimlere bakmak yeterlidir. Sonuç olarak, yüksüz fermiyonlar $3+1$ boyutlu topolojik yalıtkanların etkin eylemlerini oluşturmak için her iki biçimde de kullanılabilirler.

Çalışmanın daha sonraki kısmında kütlesiz Dirac Hamilton fonksiyonu, bu Hamilton fonksiyonunun köşegenleştirilmesi sırasında türetilen Berry ayar alanları ve onların çeşitli sistemlere uygulamaları üzerinde yoğunlaşılmıştır. Tüm çift uzay-zaman boyutlarında kütlesiz Dirac Lagrange yoğunluğunun, ayar simetrisi dışında bir de kiral simetrisi vardır. Sözkonusu kuramın elektronlarını sağ ve sol elli olarak sınıflandırmak mümkündür. Bu simetri, ayar simetrisinin belirttiği sağ ve sol elli elektronların toplamının yani elektrik akımının korunumunun dışında, klasik olarak sağ ve sol elli parçacıklara ait akımların ayrı ayrı da korunduğu anlamına gelir. Lakin bu simetri kuantizasyon sırasında regülarizasyon nedeni ile bozulmaktadır. Klasik kuram içerisinde korunan bir akımın kuantum seviyesinde korunmamasına anomali denmektedir. Kuantum kuramında akımı korunmamasına yol açan terim topolojik Chern karakteri cinsinden her çift uzay zaman boyutunda verilebilmektedir. Öte yandan son yapılan çalışmalar yarıklasik limitte de kiral anomalinin oluştuğunu göstermektedir. Dinamik sistemlerin yarı-klasik analizi tanım olarak çeşitli belirsizlikler içerir. Çünkü spin dinamiği gibi kuantum mekaniksel olgular klasik faz uzayında incelenir. Çalışmada $3+1$ ve $5+1$ boyutlu Weyl Hamilton fonksiyonları ele alınmıştır. Faz uzayında çalışılacağı için hesaplamalar diferansiyel formlarla yapılmıştır. Bu Hamilton fonksiyonları belirli kiralliğe sahip parçacıkları içermektedir. $3+1$ boyutlu Weyl Hamilton fonksiyonunun köşegenleştirilmesinden

Abelyan bir Berry ayar alanı türetilmiştir. Böylelikle Hamilton fonksiyonunun pozitif enerjili kısmı kullanılarak klasik faz uzayında, bir Weyl parçacığının elektromanyetik ve Berry ayar alanları ile etkileşimini tarif eden Hamilton 1-form yazılmıştır. Sözkonusu 1-formun dış türevi alınarak simplektik 2-form elde edilmiştir. Weyl Hamilton fonksiyonundan elde edilen Berry eğriliğinin, momentum uzayında ortaya çıkan bir tekkutbun alanı olduğu gösterilmiştir. Faz uzayının hacim-formu, hem simplektik 2-formun kuvveti hem de simplektik matrisin determinantının karekökü cinsinden tanımlanmıştır. Hareket denklemleri türetilmiştir. Kiral anomaliyi elde etmek için faz uzayı hacim-formunun her iki tanımını da kullanarak Liouville denklemine ulaşılmıştır. Bir tekkutup alanına eşit olduğu için, Berry eğriliğinin dış türevi Dirac delta fonksiyonuna eşittir. Bu nedenle faz uzayı hacmi korunmamaktadır. Uygun bir dağılım fonksiyonu kullanarak çok paçacıklı sistemlere geçiş yapılmış ve kiral anomali ifadesi tam olarak türetilmiştir. "Kiral manyetik etki"nin elde edilmesi için de faz uzayının hacmi, hacim-formunun kuvveti cinsinden açıkça yazılmış ve Lie türevi alınmıştır. Ortaya çıkan ifadeden faz uzayı elemanlarının hareket denklemleri ve simplektik matrisin determinantının karekökü elde edilmiştir. Bu sayede kiral akım tanımlanmış ve kiral manyetik etki kısmı (akımın manyetik alan yönündeki bileşeni) açıkça gösterilmiştir. Sonrasında $5+1$ boyutlu Weyl Hamilton fonksiyonu ele alınmıştır. Bu Hamilton fonksiyonunun pozitif enerjili özvektörleri bulunmuş ve onlardan Berry ayar alanları ve Berry eğrilikleri açıkça hesaplanmıştır. $3+1$ boyuttakine benzer şekilde simplektik 2-form tanımlanmıştır. Fakat $5+1$ boyutlu uzay-zamanda Weyl Hamilton fonksiyonunun köşegenleştirilmesinden türetilen Berry ayar alanları Abelyan değillerdir. Bu nedenle simplektik 2-form öncekinden farklı olarak matris değerli olma özelliğini taşımaktadır. Bu durumda faz uzayı elemanlarına ait hız ifadelerinin uygun boyutlu matrisler ile tanımlanmaları önerilmiştir. Elde edilen hareket denklemleri de matris denklemlerdir. Temel varsayım spin ve koordinat uzaylarının ayrı ayrı ele alınabileceğidir. Bu varsayıma dayanarak matris değerli hacim-formu da tanımlanmıştır. $3+1$ boyuttakine benzer şekilde hacim-formun her iki tanımı da kullanılarak Liouville denklemi elde edilmiş ve en sonunda spin uzayı üzerinde iz işlemi yapılarak $5+1$ boyutlu kiral anomali ifadesine ulaşılmıştır. Bu boyuttaki kiral manyetik akıma ulaşmak için ise matris değerli hacim-formun simplektik matrisin kuvveti cinsiden tanımının Lie türevi alındıktan sonraki hali açıkça elde edilmiş ve matris değerli hareket denkleminin izi alınarak kiral manyetik etki terimi başarıyla elde edilmiştir. $3+1$ ve $5+1$ boyutta yarıklasik kiral anomali ve kiral manyetik akım aynı formülasyon içerisinde elde edildikten sonra bu işlem tüm $d+1$ çift uzay-zaman boyutlarına genelleştirilmiştir. Bunun için yine ilk olarak matris değerli simplektirk formun genel hali yazılmış ve hareket denklemlerinin genel hali elde edilmiştir. Hacim-formun $2d+1$ boyutlu faz uzayındaki tanımı hem simplektik 2-formun kuvveti biçiminde hem de simplektik matrisin determinantının karekökü cinsinden tanımlanmıştır. $5+1$ boyuttakine benzer şekilde, her çift $d+1$ boyutta da Berry eğriliklerinin tekkutup alanı verdiği gösterilmiş ve Liouville denklemi kullanılarak $d+1$ boyutta da kiral anomalinin oluştuğu gösterilmiştir. Kiral manyetik akıma ulaşmak için hacim-formunun tanımından yola çıkarak hareket denklemlerinin ilgili kısmı türetilmiş ve kiral manyetik etki doğru biçimde ifade edilmiştir. Böylelikle hem kiral manyetik etki, hem de kiral anomali tüm çift uzay-zaman boyutlarında aynı yarıklasik kinetik kuramın çatısı altında elde edilmiştir. Her iki etkinin kaynağının da Berry eğriliğinin sonucu olan momentum uzayındaki tekkutup olduğu gösterilmiştir.

Son kısımda ise Weyl Hamilton fonksiyonu ve ondan türetilen Berry ayar alanı ile ilgili topolojik kavramlarla ilgili hesaplamalar yapılmıştır. Sözkonusu sistemler için $d+1$ boyutlu Berry ayar alanı ve fermiyon propagatörünün topolojik dönme sayısı tanımlanmış ve çeşitli özellikleri belirtilmiştir. Öncelikle $3+1$ boyutlu Weyl sistemi incelenmiştir. $3+1$ boyutlu fermiyon dönme sayısı pozitif alt uzaya izdüşüm işlemcisi cinsinden ifade edilmiş ve bu ifadenin momentum uzayındaki tekkutup alanının diverjansı olduğu gösterilmiştir. Böylelikle dönme sayısı hesaplanmış ve kirallik sayısına eşit olduğu gösterilmiştir. Öte yandan sözkonusu tekkutbun, Berry eğriliğinden elde edilenle aynı olduğu açıkça gösterilmiştir. Dönme sayısının Chern sayısına eşit olduğu da ispatlanmıştır. Sonrasında benzer argümanların geçerliliği $5+1$ boyutlu Weyl sistemi için de ispatlanmıştır. Son olarak tüm bu sonuçlar tüm çift $d+1$ boyutlu Weyl sistemlerine taşınmış ve dönme sayılarının her durum için kirallik sayısına eşit olduğu gösterilmiştir. Öte yandan dönme sayısında ortaya çıkan tekkutbun Berry eğriliğinden elde edilenle aynı olduğu ve böylelikle her boyuttaki Weyl sistemi için yarıklasik kiral anomali ve kiral manyetik etkinin varlığı ispat edilmiştir. $d+1$ boyutlu dönme sayısının Chern sayısına eşit olduğu ve kiralliğin yarıklasik limitte tekkutbun yükü olarak ortaya çıktığı gösterilmiştir. Tekkutbun ayar alanının antisimetrik tensör ayar alanı olduğu ispat edilmiştir. Sonuçta kiralliğin yarıklasik limitte de kendini topolojik kökenli tekkutup olarak gösterdiği ve yarıklasik kiral anomaliye ve kiral manyetik etkiye sabep olduğu gösterilmiştir. Söz konusu bulguların Weyl yarımetalleri için önemi tartışılmıştır.

1. INTRODUCTION

Combining quantum mechanics and special relativity Dirac, in 1928, derived the following matrix equation

$$[i\gamma^\mu \partial_\mu - m]\psi(x) = 0, \qquad \textbf{(1.1)}$$

where x^μ is the 4-vector and the Dirac gamma matrices satisfy the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \qquad \textbf{(1.2)}$$

$\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the space-time metric and the index $\mu, \nu = 0, ..., 3$ in $3 + 1$ dimensions.

Dirac equation (1.1) describes the free, relativistic, massive, spin $1/2$ particles. Its solutions $\psi(x)$, namely the Dirac bi-spinors are the representations of the Lorentz group. In its original time dependent form, (1.1) is written as

$$(\boldsymbol{\alpha} \cdot i\boldsymbol{\nabla} + \beta m)\psi = i\hbar \frac{\partial \psi}{\partial t},$$

in which $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$. Thus one introduces the Dirac Hamiltonian in momentum space with the replacement $i\boldsymbol{\nabla} \to \boldsymbol{p}$:

$$\mathcal{H} = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m. \qquad \textbf{(1.3)}$$

Dirac's $\boldsymbol{\alpha}$ and β matrices obey the following anti-commutation relations,

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij},\ \{\alpha_i, \beta\} = 0,\ \beta^2 = 1,\ i = 1, 2, 3.$$

The interaction of the Dirac particle with the electromagnetic fields is provided via the minimal coupling $\partial_\mu \to \partial_\mu - ieA_\mu$

$$[i\gamma^\mu(\partial_\mu - ieA_\mu) - m]\psi = 0. \qquad \textbf{(1.4)}$$

So that the equation (1.4) remains invariant under $U(1)$ the gauge transformations

$$\psi \to e^{i\lambda(x)}\psi,\ A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \lambda(x),$$

yielding to the conservation of the electric charge.

One can obtain (1.4) from the Dirac Lagrangian

$$\mathcal{L}(\psi, \bar{\psi}, A) = \bar{\psi}[i\gamma^\mu(\partial_\mu - ieA_\mu) - m]\psi, \tag{1.5}$$

by calculating its functional derivative with respect to the field $\bar{\psi}$. (1.5) is an Abelian gauge theory where A_μ is the $U(1)$ gauge field. One can add the kinetic term due to A_μ as

$$\mathcal{L}_G = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

to incorporate the dynamics of the photon, in which $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength.

Besides the gauge invariance, for the massless case $m = 0$ in all even dimensions, (1.5) has a chiral symmetry under the chiral transformations

$$\psi \to e^{i\theta\gamma_5}\psi, \; \bar{\psi} \to \bar{\psi}e^{i\theta\gamma_5},$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. This classical symmetry leads to the conservation of the chiral current

$$\partial_\mu j_5^\mu = 0, \; j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi.$$

However, the chiral symmetry is broken at the quantum level due the anomaly term,

$$\partial_\mu j_5^\mu = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}.$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the $3+1$ dimensional Levi-Civita tensor. The nonconservation of the classically conserved chiral current on quantum level is called as the chiral (axial) anomaly.

1.1 Purpose of Thesis

$2+1$ dimensional Dirac theory was considered to be unrealizable before the synthesis of graphene which is a 2 dimensional electronic system made out of carbon atoms. At the so-called Dirac points of the Brillouin zone, the charge carriers have a linear dispersion relation because of the nontrivial band structure. Hence they satisfy a Dirac-like equation. Haldane proposed a model for integer quantum Hall effect without a net magnetic field which is based on $2+1$ dimensional Dirac Hamiltonian. This is a time reversal breaking (TRB) model. On the other hand, Kane and Mele build a time reversal invariant (TRI) model of spin Hall effect on graphene by incorporating the spin degrees of freedom into the Haldane's model. This model is named as the quantum spin Hall insulator or $2+1$ dimensional topological insulator. Topological insulators are supposed to be a member of topological phases. By definition, they are ordinary insulators in their bulk but they have topologically protected, robust surface states which are conducting. They are classified by means of the topological invariants.

Hence, we would like to construct an effective field theory of $2+1$ dimensional topological insulator by applying the field theory methods. Our aim is to explore the applicability of the field theoretical methods to the Kane-Mele model of spin Hall insulator. We also would like to reveal relation between the 2 dimensional Chern number expressed in terms of the Berry field strength and the winding number of the free fermion propagator which corresponds to the conductivity of the spin Hall system in the effective theory. In their model, Kane and Mele predicted that the quantization of the spin Hall conductivity will be altered slightly in the presence of the Rashba spin-orbit interaction. We also construct the effective field theory in the presence of the Rashba interaction and explore its effect on the spin Hall conductivity analytically.

Qi et al. proposed a TRI model for the $4+1$ dimensional topological insulator which is generated from the Dirac Hamiltonian with the electromagnetic interactions. They provide the theories of the lower dimensional time reversal invariant topological insulators by dimensional reduction procedure. In their model, it was shown that the θ term for the axion electrodynamics would be the effective action for the $3+1$ dimensional topological insulators. We propose a model for $3+1$ dimensional TRI topological insulator strictly within $3+1$ dimensions.

Then, we focus on the semiclassical applications of the massless Dirac Hamiltonian, namely Weyl Hamiltonian. Both Son-Yamamoto and Stephanov-Yin, by modifying the phase space with the Berry field strength, provide the chiral anomaly and chiral magnetic effect (CME) in semiclassical regime in $3+1$ dimensions. The Berry curvature emerging from the diagonalization of the Weyl Hamiltonian results in a Dirac monopole located at the center of the momentum space and this monopole modifies the Hamiltonian equations of motion as well as the phase space measure. The emergence of the chiral anomaly in the presence of the Berry curvature is generalized to higher dimensions by Dwivedi and Stone. On the other hand, Loganayagam and Surowka conjectured the CME in $d+1$ dimensions. Hence, it is desirable to obtain a general formulation which provides semiclassical chiral anomaly and CME in all even dimensions. We aimed to obtain such formulation and observe whether both phenomena are linked to each other in higher dimensions. Then we study the interrelation between the topological invariants of the Weyl Hamiltonian and try to reveal the topological origin of the Dirac monopole.

1.2 Literature Review

The graphene model based on the Dirac Hamiltonian is proposed by [1, 2]. Haldane model of integer quantum Hall effect can be found in [3]. The spin Hall phase on graphene is predicted by [4]. Theoretical prediction of the topological insulators in real materials is discussed in [5] and was observed for the first time in [6]. A review of topological insulators can be found in [7–9]. $\mathbb{Z}_2$ topological classification of topological insulators is done in [10]. For an introduction to Berry phase we refer to the Berry's original article [11]. Berry phase in condensed matter physics is reviewed in [12]. Hall conductivity and its relation to the Berry phase and Chern number is explored in [13, 14]. The effective field theory of TRI topological invariants in $4+1$ dimensions and the dimensional reduction to lower dimensions are provided in [15]. In their effective theories the external gauge fields are coupled to charge and spin currents. The effect of the Rashba spin orbit interaction on the spin Hall conductivity is argued in [4]. Confirmation of the slight change of the spin Hall conductivity is discussed in [16] by numerical methods. The experimental realization of the spin Hall phase on graphene is discussed in [17], [18]. Recent proposed materials possessing the honeycomb structure of the graphene but with a strong spin-orbit coupling are listed in [19–23]. [24] provides an effective field theory of the $2+1$ dimensional spin Hall effect by coupling the third component of the spin to the external gauge fields. The effective field theory of the spin Hall effect in the presence of the Rashba interaction is discussed in terms of the Lagrangian methods in [25]. Topological field theories are discussed in [26] in the context of the high energy physics. A review of the θ term (axion electrodynamics) was done in [27]. $3+1$ dimensional TRI topological insulators are discussed in [28–30] Semiclassical chiral magnetic effect and chiral anomaly within the chiral kinetic theory is proposed in [31,32]. Modification of the phase space measure with an introduction of the Berry field strength is discussed in [33]. CME is conjectured by [35] in higher dimensions. The generalization of the semiclassical chiral anomaly is presented in [34]. In [36], the matrix valued Hamiltonians in the presence of non-Abelian Berry gauge field are discussed as a constrained system. The Fermi liquid theory and its quasiparticle excitations are given in [37]. The quantum field theory anomalies are explained with with all details in [38]. [39] discusses the importance of topological invariants in the context of condensed matter physics. Weyl semimetals are proposed in [40] and a current review on Weyl semimetals can be found in [41].

1.3 Hypothesis

Although the underlying lattice structure of the considered condensed matter systems, the field theoretic methods in continuum limit is indispensable for such materials to understand their topological properties. The topological nature of these systems necessarily dictates the topological field theories e.g. Chern-Simons action as the effective action of the external gauge fields. These properties are directly linked to the measurable quantities like the conductivity of the topological insulators. Besides, nontrivial band structures of these systems can be understood by the Berry phase arguments in which the topological invariants are constructed by. These topological invariants can be shown to be equal to the ones which are calculated using the Green functions of the related field theory.

Chiral anomaly is the nonconservation of a classically conserved current at the quantum level, hence it is supposed to be a quantum phenomena. Recent studies of the Weyl particles interacting with electromagnetic and Berry gauge fields present the chiral anomaly and chiral magnetic effect at the semiclassical level. The Dirac monopole which is the result of the Berry curvature in momentum space is demonstrated to be responsible for the semiclassical chiral anomaly and CME. We claim that the Dirac monopole structure is general in the sense that it emerges in all even space dimensions through the diagonalization of the Weyl Hamiltonian. By this way, it is possible to prove that both the chiral anomaly and CME are originated from the existence of the Dirac monopole and they can be formulated within the same theory. The chirality property of the Weyl Hamiltonian is represented by the topological property of the Dirac monopole whose charge is given by the related Chern number.

2. BERRY GAUGE FIELD AND THE RELATION BETWEEN THE CHERN NUMBER AND THE WINDING NUMBER

2.1 Purpose

Dirac's theory in $2+1$ dimensions was believed to be physically unrealizable before the advention of the graphene which is a two dimensional electronic material. At the Dirac points and in the low energy limit, the charge carriers of graphene effectively satisfy the $2+1$ dimensional Dirac equation [1, 2]. In [3] a TRB model based on the $2+1$ dimensional Dirac Hamiltonian was constructed where an integer quantum Hall effect results without a net external magnetic field. In order to formulate the spin Hall effect in graphene, Kane and Mele [4] extended the model [3] with the spin degrees of freedom and built the TRI spin Hall effect which is now known as the 2 dimensional topological insulator. By this way, [4] yields the theoretical prediction of the topological insulator phase in real materials [5]. For the first time in [6] it was reported that this new phase was observed. Topological insulators have conducting states moving at the boundary surface [7–9] nevertheless they are insulating in the bulk.

Topological invariants already arises in the the case of quantum Hall effect which is also a topological phase of matter. Hall conductance was expressed as the first Chern number in [13, 14] which is a topological invariant. On the other hand, in $2+1$ dimensions a topological field theory is constructed by integrating out the massive Dirac electrons coupled to external Abelian gauge fields in the related path integral [42–44]. The result is the Chern-Simons action whose coefficient is the winding number of the fermion propagator. This topological number is found to be equal to another topological number, namely to the first Chern number. The first Chern number results from the Berry gauge curvature [11, 12] of the quantum Hall states. Thus the quantum Hall effect can be described with a topological field theory which manifestly violates time reversal symmetry [45, 46]. The spin Hall current of the model presented in [4] in $2+1$ dimensions was derived by calculating the related Chern numbers in [47].

However, it is possible to generate a manifestly TRI theory based on the Dirac Lagrangian of massive fermions interacting with the external electromagnetic fields by means of the $4+1$ dimensional Chern-Simons action. In [15], this $4+1$ dimensional topological action was expressed as the effective topological field theory of the fundamental TRI topological insulator in $4+1$ dimensions. They purposed that all the lower dimensional theories of the TRI topological insulators can be obtained by dimensionally reducting this effective action. It was also showed that the coefficient of this topological action is the second Chern number given by the related matrix valued Berry vector fields.

In this part, by employing the Foldy-Wouthuysen transformation, it is aimed to prove the equivalence of the coefficients of the induced Chern-Simons actions with the Chern numbers.

2.2 Foldy-Wouthuysen Transformation of the Massive Dirac Hamiltonian and the Berry Gauge Field

Free and massive electron is described by the Dirac Hamiltonian

$$H = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m. \tag{2.1}$$

We deal with the d-dimensional momentum $\boldsymbol{p}$ vector whose components are denoted by $p_{\mathrm{A}};\ A = 1, \cdots, d$. One can diagonalizes (2.1) as

$$UHU^{\dagger} = E\beta, \tag{2.2}$$

where E is the total energy

$$E = \sqrt{p^2 + m^2}, \tag{2.3}$$

and U is the Foldy-Wouthuysen transformation:

$$U = \frac{\beta H + E}{\sqrt{2E(E+m)}}.$$

A pure gauge field [48] can be extracted by means of the unitary transformation U as

$$\boldsymbol{\mathcal{A}}^{U} = iU(\boldsymbol{p})\frac{\partial U^{\dagger}(\boldsymbol{p})}{\partial \boldsymbol{p}}. \tag{2.4}$$

Utilizing the (2.4), the Berry gauge field $\boldsymbol{\mathcal{A}}$ can be acquired by projection on the positive energy eigenstates of the Dirac Hamiltonian (2.1)

$$\boldsymbol{\mathcal{A}} \equiv \mathcal{I}^{+}\boldsymbol{\mathcal{A}}^{U}\mathcal{I}^{+}, \tag{2.5}$$

where $\mathcal{I}^+$ is the projection operator onto the positive energy subspace. The Berry gauge field (2.5) leads to the Berry field strength as

$$\mathcal{G}_{\rm AB} = \frac{\partial \mathcal{A}_{\rm B}}{\partial p_{\rm A}} - \frac{\partial \mathcal{A}_{\rm A}}{\partial p_{\rm B}} - i[\mathcal{A}_{\rm A}, \mathcal{A}_{\rm B}], \tag{2.6}$$

which is in general non-vanishing.

When the $2n+1$ dimensional space-time coordinates where $n = 1, 2 \cdots$, is of concern, the Berry curvature can be used to define the Chern number which is the integral of the Chern character over a compact manifold as in [49]

$$\mathcal{N}_n = \frac{1}{(4\pi)^n n!} \int_{\mathcal{M}_{2n}} d^{2n}p \; \epsilon_{\rm A_1 A_2 \cdots A_{2n}} \mathrm{tr} \left\{ \mathcal{G}_{\rm A_1 A_2} \cdots \mathcal{G}_{\rm A_{2n-1} A_{2n}} \right\}. \tag{2.7}$$

In $2+1$ dimensions the Berry gauge field is Abelian and $\mathcal{G}_{ab} = \partial \mathcal{A}_b / \partial k_a - \partial \mathcal{A}_a / \partial k_b$, where $a, b = 1, 2$, and the first Chern number is defined to be

$$\mathcal{N}_1 = \frac{1}{4\pi} \int d^2k \epsilon_{ab} \mathrm{tr} \mathcal{G}_{ab}. \tag{2.8}$$

However,in $4+1$ dimensions it is non-Abelian and the second Chern number is defined as

$$\mathcal{N}_2 = \frac{1}{32\pi^2} \int d^4p \epsilon_{ijkl} \mathrm{tr}\{\mathcal{G}_{ij}\mathcal{G}_{kl}\}, \tag{2.9}$$

where $i, j, k, l = 1, 2, 3, 4$.

2.3 Topological Field Theories and the Chern Numbers

Field theory of electrons interacting with the external Abelian gauge field A_α is described by the Dirac Lagrangian

$$\mathcal{L}\left(\psi,\bar{\psi},A\right)=\bar{\psi}\left[\gamma^{\mu}\left(p_{\mu}+A_{\mu}\right)-m\right]\psi, \tag{2.10}$$

where $\mu=0,1\cdots d$. Integrating of the fermionic degrees of freedom in the related path integral yields to the action of the external fields as

$$S[A]=-i\ln\det[i\gamma^{\mu}(\partial_{\mu}-iA_{\mu})-m]. \tag{2.11}$$

For $d=2n$, among many other terms, it leads to

$$T[A^{n+1}]=\int[dq_1]\cdots[dq_{n+1}]A^{\mu_1}(q_1)\cdots A^{\mu_{n+1}}(q_{n+1})\pi_{\mu_1\cdots\mu_{n+1}}(q_1\cdots q_{n+1}).$$

In the above expression $[dq]$ denotes the integral over the related phase space. At the first loop order

$$\pi_{\mu_1\cdots\mu_{n+1}}(q_1\cdots q_{n+1})=\int\frac{d^{2n+1}p}{(2\pi)^{2n+1}}\mathrm{tr}\{G(p)\lambda_{\mu_1}(p,p-q_1)G(p-q_1)\cdots\lambda_{\mu_{n+1}}(p+q_{n+1},p)\},$$

where $G(p)$ is the one particle Green function of the free Dirac equation and λ_μ is the photon vertex. $T[A^{n+1}]$ generates the $(2n+1)$-dimensional Chern-Simons action

$$S^{2n+1}_{eff}[A]=C_n\int d^{2n+1}x\epsilon^{\mu_1\cdots\mu_{2n+1}}A_{\mu_1}\partial_{\mu_2}A_{\mu_3}\cdots\partial_{\mu_{2n}}A_{\mu_{2n+1}}, \tag{2.12}$$

which can be taken as the topological effective action in the low energy limit. In the weak field approximation the coefficient C_n can be written as [50]

$$C_n=\frac{\epsilon^{\mu_1\nu_1\cdots\mu_n\nu_n\mu_{n+1}}}{(n+1)(2n+1)!}\partial_{(1)\nu_1}\cdots\partial_{(n)\nu_n}\pi_{\mu_1\cdots\mu_{n+1}}(q_1\cdots q_{n+1})|_{q_i=0},$$

where $\partial_{(n)\mu}\equiv\partial/\partial q^{\mu}_{n}$. The photon vertex with will be replaced by

$$\lambda_{\mu}(p,p)=-i\partial_{\mu}G^{-1}(p),$$

where $G^{-1}(p)$ is the inverse of the free fermion propagator $G(p)$ and the coefficient C_n can be expressed as the winding number of the G [50]:

$$C_n=\frac{(-i)^{n+1}\epsilon^{\mu_1\ldots\mu_{2n+1}}}{(n+1)(2n+1)!}\int\frac{d^{2n+1}p}{(2\pi)^{2n+1}}\mathrm{tr}\{[G(p)\partial_{\mu_1}G(p)^{-1}]\ldots[G(p)\partial_{\mu_{2n+1}}G(p)^{-1}]\}. \tag{2.13}$$

In terms of the Foldy-Wouthuysen transformation U, it is possible to invert (6.2) and rewrite the Dirac Hamiltonian as $H = EU^\dagger \beta U$, which is suitable to express in a projector form. Indeed, for $p^\mu = (w, \boldsymbol{p})$ the inverse of the propagator can be written as

$$G^{-1}(p) = w + (E + i\varepsilon)(P_- - P_+), \tag{2.14}$$

where the projection operators P_+ and P_-, satisfy:

$$P_+^2 = P_+,\ P_-^2 = P_-,\ P_+ + P_- = 1. \tag{2.15}$$

They are given explicitly as

$$P_+ = U^\dagger \mathcal{I}^+ U,\ \ P_- = U^\dagger \mathcal{I}^- U. \tag{2.16}$$

The operators $\mathcal{I}^\pm$

$$\mathcal{I}^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \ \mathcal{I}^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

projects to the positive and the negative energy states respectively. Now, (2.14) can easily be inverted to obtain the Green function $G(p)$ as

$$G(p) = \frac{P_+}{w - (E + i\varepsilon)} + \frac{P_-}{w + (E + i\varepsilon)}.$$

The infinitesimal parameter ε will not be indicated unless necessary. Derivative of G^{-1} with respect to $p_0 = w$ is equal to unity:

$$\frac{\partial G^{-1}(p)}{\partial w} = 1. \tag{2.17}$$

Moreover, owing to the projector form (2.14) and the energy-momentum relation (2.3), it also satisfies

$$\frac{\partial G^{-1}(p)}{\partial p_{\mathrm{A}}} = \frac{k_{\mathrm{A}}}{E}(P_- - P_+) - 2E\frac{\partial P_+}{\partial p_{\mathrm{A}}}. \tag{2.18}$$

The following relations between the projection operators P_+ and P_-

$$P_+\frac{\partial P_-}{\partial p_{\mathrm{A}}} = -\frac{\partial P_+}{\partial p_{\mathrm{A}}}P_- = \frac{\partial P_-}{\partial p_{\mathrm{A}}}P_-;\quad P_-\frac{\partial P_-}{\partial p_{\mathrm{A}}} = -P_-\frac{\partial P_+}{\partial p_{\mathrm{A}}} = \frac{\partial P_-}{\partial p_{\mathrm{A}}}P_+,$$

can easily be derived by inspecting their basic properties (2.15).

2.3.1 Relation between C_1 and $\mathcal{N}_1$

The integration of the massive Dirac fermions in $2+1$ dimensions in the related path integral with the Lagrangian (2.10) leads to the effective action

$$S_{eff}^{2+1}[A] = C_1 \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \tag{2.19}$$

where $\mu, \nu, \rho = 0, 1, 2$. (2.19) is a topological in the sense that it does not depend on the spacetime metric tensor. It is called as $2+1$ dimensional Chern-Simons action. The coefficient C_1 is given by

$$C_1 = -\frac{1}{12}\epsilon^{\mu\nu\rho} \int \frac{d^2pdw}{(2\pi)^3} \mathrm{tr}\{[G\partial_\mu G^{-1}][G\partial_\nu G^{-1}][\partial_\rho G^{-1}]\}. \tag{2.20}$$

Making use of (2.17) (2.20) can be expressed as

$$C_1 = -\frac{1}{4}\epsilon_{ab} \int \frac{d^2pdw}{(2\pi)^3} \mathrm{tr}\{G(p)^2\partial_a G(p)^{-1} G(p)\partial_b G(p)^{-1}\}.$$

By employing (2.18), it can be easily observed that the quadratic terms in p_i vanish, so that

$$\begin{aligned} C_1 &= -\frac{1}{4}\epsilon_{ab} \int \frac{d^2pdw}{(2\pi)^3} \mathrm{tr}\{(\frac{P_+}{(w-E)^2} + \frac{P_-}{(w+E)^2})(4E^2\partial_a P_+ G \partial_b P_+ \\ &\quad -2k_a[P_- - P_+]G\partial_b P_+ - 2k_b \partial_a P_+ G[P_- - P_+])\}. \end{aligned}$$

One can observe that p_a and p_b terms combine to vanish

$$2\epsilon_{ab}\mathrm{tr}\{[\frac{P_+}{(w-E)^3} - \frac{P_-}{(w+E)^3}][k_a\partial_b P_+ + k_b \partial_a P_+] = 0,$$

even before performing the w integration. After revoking the infinitesimal parameter ε, the remaining terms become

$$C_1 = -\epsilon_{ab} \int \frac{d^2pdw}{(2\pi)^3} E^2 \mathrm{tr}\{(\frac{P_+}{(w-E-i\varepsilon)^2(w+E+i\varepsilon)} + \frac{P_-}{(w-E-i\varepsilon)(w+E+i\varepsilon)^2})\partial_a P_+ \partial_b P_+\}.$$

Integration over w yields

$$C_1 = \frac{i}{8\pi^2}\epsilon_{ab} \int d^2k \mathrm{tr}\{P_+ \partial_a P_+ \partial_b P_+\}.$$

Making use of the definitions (2.16), one can easily compute that

$$\begin{aligned} \epsilon_{ab}\mathrm{tr}\{P_+\partial_a P_+ \partial_b P_+\} &= \epsilon_{ab}\mathrm{tr}\{(\mathcal{I}^+ U \partial_a U^\dagger)(\mathcal{I}^+ U \partial_b U^\dagger) + \mathcal{I}^+ \partial_a U \partial_b U^\dagger \mathcal{I}^+\} \\ &= \epsilon_{ab}\mathrm{tr}\{\mathcal{I}^+ \partial_a U \partial_b U^\dagger \mathcal{I}^+\}. \end{aligned}$$

However, plugging the field strength (2.6) of the Abelian Berry gauge field (2.5) into (2.8) yields

$$N_1 = \frac{i}{2\pi}\int d^2k\epsilon_{ab}\mathrm{tr}\{\mathcal{I}^+\partial_a U\partial_b U^\dagger \mathcal{I}^+\}.$$

Therefore, one concludes that

$$C_1 = \frac{N_1}{4\pi}. \tag{2.21}$$

2.3.2 Relation between C_2 and $\mathcal{N}_2$

In $4+1$ dimensions, the effective topological action (2.12) becomes

$$S^{4+1}_{eff}[A] = C_2\int d^5x\epsilon^{\mu\nu\rho\alpha\beta}A_\mu\partial_\nu A_\rho\partial_\alpha A_\beta,$$

where $\mu,\nu,\cdots = 0,\cdots,4$. For $n=2$ (2.13) provides C_2 as

$$C_2 = \frac{i}{3\times 5!}\int\frac{d^4pdw}{(2\pi)^5}\epsilon^{\mu\nu\rho\alpha\beta}\mathrm{tr}\{G\partial_\mu G^{-1}G\partial_\nu G^{-1}G\partial_\rho G^{-1}G\partial_\alpha G^{-1}G\partial_\beta G^{-1}\}. \tag{2.22}$$

Plugging (2.17) and (2.18) into (2.22) yields

$$C_2 = \frac{i}{3\times 4!}\epsilon_{ijkl}\int\frac{d^4pdw}{(2\pi)^5}\mathrm{tr}\{GG[\frac{p_i}{E}(P_- - P_+) - 2E\partial_i P_+]G$$
$$[\frac{p_j}{E}(P_- - P_+) - 2E\partial_j P_+]G[\frac{p_k}{E}(P_- - P_+) - 2E\partial_k P_+]G[\frac{p_l}{E}(P_- - P_+) - 2E\partial_l P_+]\}.$$

Because of the symmetry properties the terms depending on $\boldsymbol{p}$ at the second or higher order vanish. Thus, only the following terms are allowed

$$C_2 = \frac{i}{9(2\pi)^5}\epsilon_{ijkl}\int d^4pdw\mathrm{tr}\{2E^4GG\partial_iP_+G\partial_jP_+G\partial_kP_+G\partial_lP_+$$
$$-E^2\big(k_iGG[P_- - P_+]G\partial_jP_+G\partial_kP_+G\partial_lP_+ + k_jGG\partial_iP_+G[P_- - P_+]G\partial_kP_+G\partial_lP_+$$
$$+k_kGG\partial_iP_+G\partial_jP_+G[P_- - P_+]G\partial_lP_+ + k_lGG\partial_iP_+G\partial_jP_+G\partial_kP_+G[P_- - P_+]\big)\}.$$

Inspecting the symmetry aspects one can easily observe that the second and the fifth terms are summed to vanish. Likewise, one can show that gathered together the third and the fourth terms vanish by making use of the relations

$$\partial_iP_+G(P_- - P_+) = \left(\tfrac{P_+}{w+E} - \tfrac{P_-}{w-E}\right)\partial_iP_+,\ (P_- - P_+)G\partial_lP_+ = \partial_lP_+\left(\tfrac{P_+}{w+E} - \tfrac{P_-}{w-E}\right).$$

Hence, one obtains

$$C_2 = \frac{2i}{9(2\pi)^5}\epsilon_{ijkl}\int d^4pdwE^4\mathrm{tr}[G^2\partial_iP_+G\partial_jP_+G\partial_kP_+G\partial_lP_+].$$

After recalling the ε by the replacement $E \to E + i\varepsilon$, this can be expressed as

$$C_2 = \frac{2i}{9(2\pi)^5}\epsilon_{ijkl}\int d^p k dw \quad E^4 \mathrm{tr}[(\tfrac{P_+}{(w-E-i\varepsilon)^3(w+E+i\varepsilon)^2} + \tfrac{P_-}{(w-E-i\varepsilon)^2(w+E+i\varepsilon)^3}) \\ \partial_i P_+ \partial_j P_+ \partial_k P_+ \partial_l P_+].$$

By performing the w integration $\boldsymbol{C_2}$ is calculated to be

$$C_2 = -\frac{1}{12}\epsilon_{ijkl}\int \frac{d^4 p}{(2\pi)^4}\mathrm{tr}\{\partial_i P_+ P_- \partial_j P_+ \partial_k P_+ P_- \partial_l P_+\}. \tag{2.23}$$

In terms of the explicit forms of P_- and P_+ given by (2.16) the integrand of (2.23) can be expressed as

$$\begin{aligned} & \epsilon_{ijkl}\mathrm{tr}\{\partial_i P_+ P_- \partial_j P_+ \partial_k P_+ P_- \partial_l P_+\} \\ = \ & \epsilon_{ijkl}\mathrm{tr}\{(U^\dagger \mathcal{I}^+ \partial_i U + \partial_i U^\dagger \mathcal{I}^+ U) P_- (U^\dagger \mathcal{I}^+ \partial_j U + \partial_j U^\dagger \mathcal{I}^+ U) \\ & (U^\dagger \mathcal{I}^+ \partial_k U + \partial_k U^\dagger \mathcal{I}^+ U) P_- (U^\dagger \mathcal{I}^+ \partial_l U + \partial_l U^\dagger \mathcal{I}^+ U)\} \\ = \ & \epsilon_{ijkl}\mathrm{tr}\{\mathcal{I}^+ \partial_i U P_- \partial_j U^\dagger \mathcal{I}^+ \partial_k U P_- \partial_l U^\dagger \mathcal{I}^+\}. \end{aligned} \tag{2.24}$$

The Berry gauge field (2.5) is non-Abelian and its curvature $\mathcal{F}_{ij}$ can be written as

$$\begin{aligned} \mathcal{F}_{ij} &= i\mathcal{I}^+ \partial_i U \partial_j U^\dagger \mathcal{I}^+ + i\mathcal{I}^+ U \partial_i U^\dagger \mathcal{I}^+ U \partial_j U^\dagger \mathcal{I}^+ - i \leftrightarrow j \\ &= i\mathcal{I}^+ \partial_i U P_- \partial_j U^\dagger \mathcal{I}^+ - i\mathcal{I}^+ \partial_j U P_- \partial_i U^\dagger \mathcal{I}^+. \end{aligned}$$

Restoring it into the definition of the second Chern number (2.9) and inspecting (2.23) and (2.24) one comes to the conclusion

$$C_2 = \frac{N_2}{24\pi^2}. \tag{2.25}$$

Generalizing this method to higher dimensions is quite straightforward.

2.4 Discussions

We use the Foldy-Wouthuysen transformation of Dirac Hamiltonians in order to acquire the Berry gauge fields. Explicitly, the first and second Chern numbers are derived. It is shown that the winding numbers of the free Dirac propagators are equal to the coefficients of the effective Chern-Simons actions in both $2 + 1$ and $4 + 1$ dimensions. This construction can be generalized to higher dimensions straightforwardly.

3. EFFECTIVE FIELD THEORY OF $2+1$ DIMENSIONAL TOPOLOGICAL INSULATOR IN THE PRESENCE OF RASHBA SPIN-ORBIT COUPLING

3.1 Purpose

The Kane-Mele model of monolayer graphene [4] provides a formulation of the 2+1 dimensional time reversal invariant topological insulator which is also known as the quantum spin Hall insulator. In this new topological phase of matter the bulk is insulating but there exist topologically protected gapless states moving on the edge. In the low energy and long wavelength regime the charge carriers of graphene are effectively massless Dirac-like fermions at the Dirac points K, K'. In [4] properties of these electrons in the presence of intrinsic as well as Rashba spin-orbit interactions are investigated. In the Kane-Mele model when only the intrinsic spin-orbit coupling is considered, two copies of the Haldane model [3] are combined to provide a quantized (integer) spin Hall conductivity. Kane and Mele also argued that when both the intrinsic and Rashba spin-orbit coupling terms are present, although the spin Hall conductivity is not quantized it has a value which slightly differs from the quantized one. Actually, this is confirmed by [16] using numerical methods. However, it is reported in [17], [18] that the spin Hall phase is not experimentally realizable in graphene because of the weak intrinsic spin-orbit coupling strength. Nevertheless, there are some recent proposals of synthesizing new materials which possess the honeycomb structure of graphene with a large spin-orbit gap [19–23].

An efficient tool to reveal the general predictions of topological insulators is to construct the effective field theory of the external fields coupled to charge and spin degrees of freedom [15]. Effective theories are insensible to the internal structure of the inspected material. They give its response to the external fields. When only the intrinsic spin-orbit coupling term is taken into account, the effective theory of $2+1$ dimensional TRI topological insulator is well established in [15, 52]. This is a topological field theory where the coefficients appearing in the action are related to the first Chern numbers of the constituting Dirac-like Hamiltonians. One can couple an external field to the third component of spin s_z which combines with the electromagnetic field to procure a non-vanishing effective action whose coefficient is the quantized spin Hall conductivity [24]. The third component of spin is not conserved in the presence of Rashba interaction, nevertheless one can still deal with the spin Hall conductivity whose presence indicates the spin Hall phase.

The aim is to explore whether the expected spin Hall conductivity can be obtained from the effective action of the external electromagnetic and spin fields within the model [4] in the presence of Rashba interaction. This effective theory was studied in terms of Lagrangian methods in [25]. However in this work the Hamiltonian methods are adopted to derive the effective action of the external fields coupled to charge and spin degrees of freedom of the full Kane-Mele model where the link between the coefficients taking part in the effective theory and topological Chern numbers can be found straightforwardly.

3.2 Model and the Effective Field Theory

Graphene possesses a honeycomb lattice structure based on the A and B sublattices. At the two inequivalent Dirac points $K,\ K'$ of the Brillouin zone valence and conduction bands touch each other yielding conductivity. Around these points in the low energy and long wavelength limit charge carriers effectively obey the free, massless Dirac-like Hamiltonian

$$H_0 = \sigma_x \tau_z p_x + \sigma_y p_y,$$

where the effective velocity of electrons is set as $v_F = 1$. The Pauli spin matrices $\sigma_{x,y,z}$ act on the states of the sublattices $A,\ B$, while $\tau_z = \mathrm{diag}(1, -1)$ indicates the Dirac points $K,\ K'$. In [4] it is suggested to generate a mass gap by the intrinsic spin-orbit coupling term

$$H_{SO} = \Delta_{SO} \sigma_z \tau_z s_z.$$

They also considered the Rashba spin-orbit interaction term

$$H_{\mathrm{R}} = \lambda_{\mathrm{R}} (\sigma_x \tau_z s_y - \sigma_y s_x).$$

In the above expression the constant parameter λ_{R} is experimentally tunable. The Pauli spin matrices $s_{x,y,z}$ correspond to the spin degrees of freedom of chatge carriers. Thus, the total Hamiltonian of the Kane-Mele model is

$$H = H_0 + H_{SO} + H_{\mathrm{R}}. \tag{3.1}$$

For $\lambda_{\mathrm{R}} = 0$ the third component of spin, s_z, which can be labeled by $\uparrow\downarrow$ is conserved. Hence, spin current can directly be defined as $j^{spin} = j^{\uparrow} - j^{\downarrow}$. It leads to the quantized spin Hall conductivity, that is $\sigma_{SH} = 1/2\pi$, in the $e = 1,\ \hbar = 1$ units. This spin current can also be derived from the action, $\mu, \nu, \rho = 0, 1, 2,$

$$S_s = \frac{1}{2\pi} \int d^3x \epsilon^{\mu\nu\rho} \Omega_\mu \partial_\nu A_\rho, \tag{3.2}$$

where A_μ and Ω_μ are the external electromagnetic and spin gauge fields currents [24]. This result (3.2) is obtained as the effective action which is computed by integrating out the fermionic fields in the path integral of the field theory described by the following Lagrangian density of the Kane-Mele model for $\lambda_{\mathrm{R}} = 0$,

$$\mathcal{L}_0 = \bar{\psi} \left[\gamma^\mu \left(i\partial_\mu + A_\mu + \frac{S_z}{2} \Omega_\mu \right) - \Delta_{SO} \right] \psi, \tag{3.3}$$

where

$$\gamma^0 = \sigma_z \tau_z s_z,\ \gamma^1 = i\sigma_y s_z,\ \gamma^2 = -i\sigma_x \tau_z s_z,$$

and $S_z = \mathrm{diag}\left(s_z, s_z, s_z, s_z\right)$. It is desirable to extend this stratagem for deriving the spin Hall conductivity to the fully fledged Kane-Mele model given by (3.1). Although when λ_{R} is nonvanishing the third component of spin does not commute with the Hamiltonian (3.1), so that the current $j^s_\mu = \bar{\psi}\gamma_\mu \frac{S_z}{2} \psi$ is not conserved, one can still define a conserved spin current [53] and calculate the spin Hall conductivity. Indeed, Kane and Mele argued that when $\Delta_{SO} > \lambda_{\mathrm{R}}$ the Hamiltonian (3.1) yields the spin Hall conductivity which slightly differs from the quantized value $1/2\pi$. This is confirmed in [16] by studying the model numerically. The goal of this work is to derive the effective field theory of external fields A_μ, Ω_μ, considering the Kane-Mele model Lagrange density in the presence of Rashba spin-orbit interaction:

$$\mathcal{L}\left(\psi, \bar{\psi}, A, \Omega\right) = \bar{\psi} \left[\gamma^\mu \left(i\partial_\mu + A_\mu + \frac{S_z}{2} \Omega_\mu \right) - \Delta_{SO} - \lambda_R (\sigma_y s_x - \sigma_x \tau_z s_y) \right] \psi \tag{3.4}$$

In the partition function,

$$Z = \int D\psi D\bar{\psi} D A_\mu D\Omega_\mu e^{i \int d^3x \mathcal{L}},$$

one may integrate out ψ and $\bar{\psi}$ to acquire the effective theory of the external fields A_μ, Ω_μ:

$$\int D\psi D\bar{\psi} DA_\mu D\Omega_\mu e^{iS} = \int DA_\mu D\Omega_\mu e^{iS_{eff}}.$$

S_{eff} is defined as

$$S_{eff}[A,\Omega] = -i \ln \det \left[i\gamma^\mu(\partial_\mu - iA_\mu - i\frac{S_z}{2}\Omega_\mu) - \Delta_{SO} - \lambda_{\mathrm{R}}(\sigma_y s_x - \sigma_x \tau_z s_y) \right] \quad \textbf{(3.5)}$$

In the low energy limit one can interest only in the following topological terms which (3.5) evokes ,

$$S_{eff} = C\int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + C_s \int d^3x \epsilon^{\mu\nu\rho} \Omega_\mu \partial_\nu A_\rho + C_\Omega \int d^3x \epsilon^{\mu\nu\rho} \Omega_\mu \partial_\nu \Omega_\rho. \quad \textbf{(3.6)}$$

(3.6) yields the spin current

$$j^\mu_{spin} = \frac{\delta S_{eff}}{\delta \Omega_\mu}$$

It is worth mentioning that the spin current obtained from the action (3.6) is conserved, that is $\partial_\mu j^\mu_{spin} = 0$, though the third component of spin S_z does not commute with the Hamiltonian (3.1). It is a consequence of dealing with the low-energy limit where the higher order gradient terms in the expansion of (3.5) are ignored. In the weak field approximation the coefficients in (3.6) are given in terms of the fermion propagator $G(p)$ and its inverse $G^{-1}(p)$ by [50]

$$C = -\frac{1}{12}\epsilon^{\mu\nu\rho} \int \frac{d^3p}{(2\pi)^3} \mathrm{tr}\left\{ \left[G\partial_\mu G^{-1}\right] \left[G\partial_\nu G^{-1}\right] \left[G\partial_\rho G^{-1}\right] \right\}, \quad \textbf{(3.7)}$$

$$C_s = -\frac{1}{12}\epsilon^{\mu\nu\rho} \int \frac{d^3p}{(2\pi)^3} \mathrm{tr}\left\{ S_z \left[G\partial_\mu G^{-1}\right] \left[G\partial_\nu G^{-1}\right] \left[G\partial_\rho G^{-1}\right] \right\}, \quad \textbf{(3.8)}$$

$$C_\Omega = -\frac{1}{12}\epsilon^{\mu\nu\rho} \int \frac{d^3p}{(2\pi)^3} \mathrm{tr}\left\{ S_z \left[G\partial_\mu G^{-1}\right] S_z \left[G\partial_\nu G^{-1}\right] \left[G\partial_\rho G^{-1}\right] \right\}, \quad \textbf{(3.9)}$$

where $\partial_\mu \equiv \partial/\partial p^\mu$. For $\lambda_{\mathrm{R}} = 0$, one can write $H_0 + H_{SO} = \mathrm{diag}(H^{\uparrow +}, H^{\uparrow -}, H^{\downarrow +}, H^{\downarrow +})$ in terms of 2×2 matrices, where $\pm$ labels the Dirac points $K,\ K'$. One can show that the coefficients can be expressed in terms of the related first Chern numbers ([15], [52] and the references therein). Indeed, the coefficients of the Chern-Simons terms are given by

$$C(\lambda_{\mathrm{R}} = 0) = C_\Omega(\lambda_{\mathrm{R}} = 0) = (N_1^{\uparrow +} + N_1^{\uparrow -} + N_1^{\downarrow +} + N_1^{\downarrow +})/4\pi.$$

The related first Chern numbers were obtained to be $N_1^{\uparrow \pm} = 1/2,\ N_1^{\downarrow \pm} = -1/2$. Thereby one observes that $C(\lambda_{\mathrm{R}} = 0) = C_\Omega(\lambda_{\mathrm{R}} = 0) = 0$. Vanishing of these coefficients was expected due to the fact that Kane-Mele model is time reversal invariant but the Chern-Simons terms lack this symmetry. However, the other coefficient is given by

$$C_s(\lambda_{\rm R}=0) = (N_1^{\uparrow +} + N_1^{\uparrow -} - N_1^{\downarrow +} - N_1^{\downarrow +})/4\pi = 1/2\pi.$$

Therefore, (3.2) occurs to be the effective action of the theory described by (3.3). In the presence of Rashba interaction, i.e. $\lambda_{\rm R} \neq 0$, the coefficients (3.7)-(3.9) were constructed within Lagrangian methods in [25] up to the first-order terms in λ_R/Δ_{SO}. Moreover, in [54] an effective theory for a model which is similar to the Kane-Mele model was constructed. In [54], it was claimed that the model considered is equivalent to the Kane-Mele model (3.1), because of only employing another representation of the gamma matrices given by $\gamma^\mu = \sigma_z \gamma^\mu_{KM}$, where γ^μ_{KM} are the gamma matrices of the Kane-Mele model. However, they adopted the definition $\gamma^0 = \sigma_z$ which leads to the erroneous result $\gamma^0_{KM} = 1$. The following section is devoted to calculate these coefficients explicitly.

3.3 Calculation of the Coefficients

In order to attain the one particle Green function of free Dirac field $G(p)$, one would like to employ the Hamiltonian methods. The matrix form of the Kane-Mele model Hamiltonian (3.1) can be written as

$$H = \begin{pmatrix} \Delta_{SO} & 0 & 0 & 0 & p_x - ip_y & 0 & 0 & 0 \\ 0 & -\Delta_{SO} & 0 & 0 & 2i\lambda_{\rm R} & p_x - ip_y & 0 & 0 \\ 0 & 0 & -\Delta_{SO} & 0 & 0 & 0 & -p_x - ip_y & 2i\lambda_{\rm R} \\ 0 & 0 & 0 & \Delta_{SO} & 0 & 0 & 0 & -p_x - ip_y \\ p_x + ip_y & -2i\lambda_{\rm R} & 0 & 0 & -\Delta_{SO} & 0 & 0 & 0 \\ 0 & p_x + ip_y & 0 & 0 & 0 & \Delta_{SO} & 0 & 0 \\ 0 & 0 & -p_x + ip_y & 0 & 0 & 0 & \Delta_{SO} & 0 \\ 0 & 0 & -2i\lambda_{\rm R} & -p_x + ip_y & 0 & 0 & 0 & -\Delta_{SO} \end{pmatrix}. \tag{3.10}$$

In terms of $p^2 = p_x^2 + p_y^2$, the eigenvalues of (3.10) are calculated to be

$$\begin{aligned} E_1 = E_2 = &\ \lambda_R + \sqrt{(\Delta_{SO} - \lambda_R)^2 + p^2}, \\ E_3 = E_4 = &\ -\lambda_R + \sqrt{(\Delta_{SO} + \lambda_R)^2 + p^2}, \\ E_5 = E_6 = &\ \lambda_R - \sqrt{(\Delta_{SO} - \lambda_R)^2 + p^2}, \\ E_7 = E_8 = &\ -\lambda_R - \sqrt{(\Delta_{SO} + \lambda_R)^2 + p^2}. \end{aligned} \tag{3.11}$$

$G(p)$ can be acquired by means of the Foldy-Wouthuysen unitary transformation U which is defined to satisfy

$$UHU^{\dagger} = \mathrm{diag}(E_1, \cdots, E_8) \equiv \sum_{M=1}^{8} E_M I^M. \qquad \textbf{(3.12)}$$

Here I^M is the matrix whose elements vanish other than $(I^M)_{MM} = 1$. The eigenfunctions corresponding to the energy eigenvalues (3.11) can be employed to establish the unitary matrix U which diagonalizes the Hamiltonian (3.10) as

$$U = \begin{pmatrix} 0 & 0 & -iF_1 & \frac{(p_x-ip_y)F_1}{\Delta_{SO}-E_1} & 0 & 0 & \frac{-i(p_x+ip_y)F_1}{\Delta_{SO}-E_1} & F_1 \\ \frac{i(p_x+ip_y)F_2}{p_x-ip_y} & \frac{-(\Delta_{SO}-E_2)F_2}{p_x-ip_y} & 0 & 0 & \frac{-i(\Delta_{SO}-E_2)F_2}{p_x-ip_y} & F_2 & 0 & 0 \\ 0 & 0 & iF_3 & \frac{(p_x-ip_y)F_3}{\Delta_{SO}-E_3} & 0 & 0 & \frac{i(p_x+ip_y)F_3}{\Delta_{SO}-E_3} & F_3 \\ \frac{-i(p_x+ip_y)F_4}{p_x-ip_y} & \frac{-(\Delta_{SO}-E_4)F_4}{p_x-ip_y} & 0 & 0 & \frac{i(\Delta_{SO}-E_4)F_4}{p_x-ip_y} & F_2 & 0 & 0 \\ 0 & 0 & -iF_5 & \frac{(p_x-ip_y)F_5}{\Delta_{SO}-E_5} & 0 & 0 & \frac{-i(p_x+ip_y)F_5}{\Delta_{SO}-E_5} & F_5 \\ \frac{i(p_x+ip_y)F_6}{p_x-ip_y} & \frac{-(\Delta_{SO}-E_6)F_6}{p_x-ip_y} & 0 & 0 & \frac{-i(\Delta_{SO}-E_6)F_6}{p_x-ip_y} & F_6 & 0 & 0 \\ 0 & 0 & iF_7 & \frac{(p_x-ip_y)F_7}{\Delta_{SO}-E_7} & 0 & 0 & \frac{i(p_x+ip_y)F_7}{\Delta_{SO}-E_7} & F_7 \\ \frac{-i(p_x+ip_y)F_8}{p_x-ip_y} & \frac{-(\Delta_{SO}-E_8)F_8}{p_x-ip_y} & 0 & 0 & \frac{i(\Delta_{SO}-E_8)F_8}{p_x-ip_y} & F_8 & 0 & 0 \end{pmatrix}. \qquad \textbf{(3.13)}$$

The normalization factors are given by

$$F_{2m-1} = \sqrt{\frac{(\Delta_{SO} - E_m)^2}{2((\Delta_{SO} - E_m)^2 + p^2)}},\ F_{2m} = \sqrt{\frac{p^2}{2((\Delta_{SO} - E_m)^2 + p^2)}} \qquad \textbf{(3.14)}$$

where $m = 1, 2, 3, 4$. (3.14) satisfy the following condition:

$$F_{2m-1}^2 + F_{2m}^2 = \frac{1}{2}. \qquad \textbf{(3.15)}$$

Inverting the unitary transformation (3.12), it is possible to retrieve the Hamiltonian (3.10) in the form

$$H = \sum_{M=1}^{8} E_M P^M,$$

where $P^M = U^{\dagger} I^M U$. is introduced. Obviously P^M are projection operators:

$$\sum_{M=1}^{8} P^M = 1,\ \ P^M P^N = \delta^{MN} P^N.$$

Now, one can construct the Green function $G(p)$ and its inverse $G^{-1}(p)$ as

$$G(p) = \sum_{M=1}^{8} \frac{P^M}{w - E_M},\ G^{-1}(p) = w - \sum_{M=1}^{8} E_M P^M, \qquad \textbf{(3.16)}$$

where $p^\mu = (w, p_a);\ a = 1, 2$. Note that the derivatives of the inverse Green function $G^{-1}(p)$ obey

$$\frac{\partial G^{-1}(p)}{\partial w} = 1,\ \frac{\partial G^{-1}(p)}{\partial p_a} = -\sum_{M=1}^{8} \left(\frac{p_a}{E_M} P^M + E_M \partial_a P^M \right). \qquad \textbf{(3.17)}$$

To proceed we would like to perform the w integrations in (3.7)-(3.9). This requires that the energies $E_M;\ M = 1, \cdots, 8$, are arranged to be definitely positive or negative. One restricts the values of coupling constant to $\Delta_{SO} > 2\lambda_{\mathrm{R}}$, so that it is possible to divide the spectrum as $E_\alpha,\ \alpha = 1, 2, 3, 4$, which are positive and $E_i,\ i = 5, 6, 7, 8$, which are negative. Notice that in the limit $\lambda_R \to 0$, the eigenvalues $E_{1,2}$ and $E_{3,4}$ (similarly $E_{5,6}$ and $E_{7,8}$) approach each other. Hereafter the following conventions are adopted: $\alpha, \beta, \gamma = 1, ..., 4$; $i, j, k = 5, ..., 8$; $M, N = 1, ..., 8$; and $a, b = 1, 2$.

3.3.1 Calculation of C and C_Ω

The winding number (3.7) can be written as

$$C = -\frac{1}{4}\varepsilon^{ab} \int \frac{d^2 p dw}{(2\pi)^3} \mathrm{tr} \left\{ G^2(p) \partial_a G^{-1}(p) G(p) \partial_b G^{-1}(p) \right\}. \qquad \textbf{(3.18)}$$

The repeating a and b indices are summed over. Employing the definition of the Green function (3.16) and the derivatives of its inverse given in (3.17) one can demonstrate that the terms explicitly linear and quadratic in p_a vanish directly and after performing the w integration the remaining part leads to

$$C = \frac{-i\varepsilon^{ab}}{16\pi^2} \int d^2 p$$
$$\mathrm{tr} \left\{ \sum_{\alpha,\beta,i} \frac{E_\beta P^i \left(\partial_a P^\alpha \partial_b P^\beta - \partial_b P^\beta \partial_a P^\alpha \right)}{E_\alpha - E_i} + \sum_{\alpha,i,j} \frac{E_j P^\alpha \left(\partial_a P^i \partial_b P^j - \partial_b P^j \partial_a P^i \right)}{E_\alpha - E_i} \right\}.$$

The terms other than $\alpha = \beta$ and $i = j$ do not contribute, hence one gets

$$C = -\frac{i}{8\pi^2}\varepsilon^{ab} \sum_{\alpha,i} \int d^2 p \mathrm{tr} \left\{ \frac{E_\alpha}{E_\alpha - E_i} P^i \partial_a P^\alpha \partial_b P^\alpha + \frac{E_i}{E_\alpha - E_i} P^\alpha \partial_a P^i \partial_b P^i \right\}. \qquad \textbf{(3.19)}$$

Remembering that $P^M = U^\dagger I^M U$, (3.19) can be expressed in the form

$$C = \frac{i}{8\pi^2}\varepsilon^{ab} \sum_{\alpha,i} \int d^2 p \mathrm{tr} \left\{ I^\alpha A_a^U I^i A_b^U \right\}, \qquad \textbf{(3.20)}$$

where $A_a^U = iU\partial_a U^\dagger$ is introduced. Because of being a pure gauge field curvature of A_a^U identically vanishes. However, one can construct the Berry gauge field through the projection of the A^U to the positive energy states [48] by

$$A^B_a = i\sum_{\alpha,\beta} I^\alpha U\partial_a U^\dagger I^\beta,$$

whose field strength

$$F^B_{ab} = \partial_a A^B_b - \partial_b A^B_a - i[A^B_a, A^B_a],$$

does not vanish in general. The first Chern number is defined in terms of the Berry curvature as

$$N_1 = \frac{1}{4\pi}\int d^2p\varepsilon^{ab}\mathrm{tr}F^B_{ab}. \tag{3.21}$$

The topological numbers (3.18) and (3.21) are connected to each other by

$$C = \frac{N_1}{4\pi}.$$

This relation can be accomplished by observing that, due to the identity $\sum_i I^i = 1 - \sum_\alpha I^\alpha$, one can express (3.20) as

$$C = \frac{-i\varepsilon^{ab}}{8\pi^2}\sum_\alpha\int d^2p\mathrm{tr}\left\{I^\alpha U\partial_a U^\dagger U\partial_b U^\dagger I^\alpha\right\}+\frac{i\varepsilon^{ab}}{8\pi^2}\sum_{\alpha,\beta}\int d^2p\mathrm{tr}\left\{I^\alpha U\partial_a U^\dagger I^\beta U\partial_b U^\dagger I^\alpha\right\}. \tag{3.22}$$

Having attained the relation between the Chern number (3.21) and the winding number (3.18), the next step is to calculate the coefficient C explicitly. In a straightforward manner (3.22) can be written as

$$C = \frac{i}{8\pi^2}\varepsilon^{ab}\int \mathrm{tr}\left\{\sum_\alpha I^\alpha\partial_a U\partial_b U^\dagger I^\alpha - \sum_{\alpha,\beta} I^\alpha\partial_a UP^\beta\partial_b U^\dagger I^\alpha\right\}. \tag{3.23}$$

After performing the trace operation and making use of (3.13), the first term vanishes:

$$\partial_a U_{21}\partial_b U^\star_{21} + 2\partial_a U_{22}\partial_b U^\star_{22} + \partial_a U_{41}\partial_b U^\star_{41} + 2\partial_a U_{42}\partial_b U^\star_{42} = 0.$$

On the other hand, the second term in (3.23) yields

$$\begin{aligned}
\varepsilon^{ab}\big\{&(P^2_{11}+P^4_{11})\left(\partial_a U_{21}\partial_b U^\star_{21} + \partial_a U_{41}\partial_b U^\star_{41}\right)\\
&+4i\mathrm{Im}\left[P^2_{12}\left(\partial_a U_{21}\partial_b U^\star_{22} + \partial_a U_{25}\partial_b U^\star_{26}\right) + P^4_{12}\left(\partial_a U_{41}\partial_b U^\star_{42} + \partial_a U_{45}\partial_b U^\star_{46}\right)\right]\\
&+2i\mathrm{Im}[(P^2_{16}+P^4_{16})\left(\partial_a U_{21}\partial_b U^\star_{26} + \partial_a U_{41}\partial_b U^\star_{46}\right)]\\
&+4P^2_{22}\partial_a U_{22}\partial_b U^\star_{22} + 4P^4_{22}\partial_a U_{42}\partial_b U^\star_{42}\big\}.
\end{aligned} \tag{3.24}$$

In terms of the polar coordinates

$$p = \sqrt{p_x^2 + p_y^2},\ \theta = \arctan\frac{p_y}{p_x}, \tag{3.25}$$

one can demonstrate that (3.24) vanishes as

$$\frac{4i}{p}(F_2^2\partial_pF_2^2+F_4^2\partial_pF_4^2)-\frac{8i}{p}(F_2^3\partial_pF_2+F_4^3\partial_pF_4)]=0.$$

In these calculations the explicit forms of the Green functions obtained from the Kane-Mele model are used. However, the properties of Green functions which led us to conclude that the coefficient C vanishes, are extendable to any Dirac-like theory whose energy spectrum possesses particle-hole (antiparticle) symmetry.

One of the benefits of using Hamiltonian methods is the fact that the coefficients corresponding to the subspaces labeled by $\tau_z = \pm 1$ and $s_z = \uparrow\downarrow$ can be calculated explicitly. In fact, they yield the Chern numbers

$$N_1^{\uparrow\pm}=1/2,\ N_1^{\downarrow\pm}=1/2.$$

The coefficient C_Ω (3.9), can be demonstrated to be equal to C (3.7). Observe that

$$S_zH(\lambda_{\rm R})S_z=H(-\lambda_{\rm R}).$$

This interchanges the positive and negative energy eigenvalues within themselves: $E_1 \leftrightarrow E_3$ and $E_5 \leftrightarrow E_7$. Thereby $S_zG^{-1}S_z$ and S_zGS_z are captured from G^{-1} and G by this exchange of eigenvalues. However, the initial w integrals which led to (3.19) are not altered under the exchange $E_1 \leftrightarrow E_3$ and $E_5 \leftrightarrow E_7$. Therefore one concludes that $C_\Omega = C = 0$.

Vanishing of the coefficients C_Ω and C, was expected because of the fact that under the time reversal symmetry Kane-Mele model is invariant but Chern-Simons action acquires an overall minus sign.

3.3.2 Calculation of C_s

Substituting the first derivative of the inverse Green function with respect to w by (3.17), the coefficient C_s, (3.8), can be separated into three parts:

$$\begin{aligned}C_s &= -\frac{1}{12}\varepsilon^{ab}\int\frac{d^2pdw}{(2\pi)^3}\mathrm{tr}\Big\{S_zG^2\partial_aG^{-1}G\partial_bG^{-1}-S_zG\partial_aG^{-1}G^2\partial_bG^{-1}\\ &\quad +S_zG\partial_aG^{-1}G\partial_bG^{-1}G\Big\}\equiv C_s^{(1)}+C_s^{(2)}+C_s^{(3)}.\end{aligned}\qquad \textbf{(3.26)}$$

Restoring $\partial_aG^{-1}(p)$ given by (3.17) into (6.15) yields terms which are explicitly linear and quadratic in p_a. Obviously, the terms quadratic in p_a vanish. After some calculations one can show on general grounds that the terms linear in p_a also yield a vanishing contribution. In order to deal with the remaining terms,firstly one would like to perform the w integration. For this aim, first and the second constituents of (6.15) can be written as

$$C_s^{(1)} = -\frac{\varepsilon^{ab}}{12}\sum_{\alpha,i,M}\int\frac{d^2pdw}{(2\pi)^3}\mathrm{tr}\left\{\left(\frac{S_zE_iP^\alpha\partial_aP^i}{(w-E_\alpha)^2(w-E_i)} - \frac{S_zE_iP^i\partial_aP^\alpha}{(w-E_i)^2(w-E_\alpha)}\right)E_M\partial_bP^M\right\},$$
$$C_s^{(2)} = \frac{\varepsilon^{ab}}{12}\sum_{\alpha,i,M}\int\frac{d^2pdw}{(2\pi)^3}\mathrm{tr}\left\{\left(\frac{S_zE_iP^\alpha\partial_aP^i}{(w-E_\alpha)(w-E_i)^2} - \frac{S_zE_iP^i\partial_aP^\alpha}{(w-E_i)(w-E_\alpha)^2}\right)E_M\partial_bP^M\right\}.$$

Now one can integrate over w and find that they acquire the same form:

$$C_s^{(1)} = C_s^{(2)} = -\frac{i}{48\pi^2}\varepsilon^{ab}\sum_{\alpha,i,M}\int d^2p\mathrm{tr}\Big\{S_z\frac{P^\alpha\partial_aP^i + P^i\partial_aP^\alpha}{E_\alpha - E_i}E_M\partial_bP^M\Big\}.$$

They can be expressed as

$$C_s^{(1)} = C_s^{(2)} = -\frac{i}{48\pi^2}\sum_{\alpha,i,M}\int d^2p\mathrm{tr}\left\{\frac{(E_i-E_M)P^MS_zP^\alpha\mathcal{I}^i + (E_\alpha - E_M)P^MS_zP^i\mathcal{I}^\alpha}{E_\alpha - E_i}\right\}, \tag{3.27}$$

where the definition $\mathcal{I}^M = \varepsilon^{ab}\partial_aU^\dagger I^M\partial_bU$ is used. However, due to the fact that $[S_z, P^M] \neq 0$, the third constituent of (6.15) yields

$$C_s^{(3)} = -\frac{\varepsilon^{ab}}{12}\sum_{L,M\neq N}\int\frac{d^2pd\omega}{(2\pi)^3}\mathrm{tr}\left\{\frac{(E_N-E_M)P^LS_zP^M(E_L\partial_aP^N\partial_bP^L + E_N\partial_aP^N\partial_bP^N)}{(\omega-E_L)(\omega-E_M)(\omega-E_N)}\right\}.$$

By performing the w integration one obtains

$$\begin{aligned} C_s^{(3)} = & -\frac{i}{48\pi^2}\int d^2p\mathrm{tr}\Big\{P^{57}\left(\mathcal{I}^1+\mathcal{I}^3\right) + P^{68}\left(\mathcal{I}^2+\mathcal{I}^4\right) - P^{13}\left(\mathcal{I}^5+\mathcal{I}^7\right) - P^{24}\left(\mathcal{I}^6+\mathcal{I}^8\right) \\ & + P^{35}\left(\frac{E_1-E_3}{E_5-E_3}\mathcal{I}^1 - \frac{E_7-E_5}{E_3-E_5}\mathcal{I}^7\right) + P^{46}\left(\frac{E_2-E_4}{E_6-E_4}\mathcal{I}^2 - \frac{E_8-E_6}{E_4-E_6}\mathcal{I}^8\right) \\ & + P^{17}\left(\frac{E_3-E_1}{E_7-E_1}\mathcal{I}^3 - \frac{E_5-E_7}{E_1-E_7}\mathcal{I}^5\right) + P^{28}\left(\frac{E_4-E_2}{E_8-E_2}\mathcal{I}^4 - \frac{E_6-E_8}{E_2-E_8}\mathcal{I}^6\right)\Big\}, \end{aligned} \tag{3.28}$$

where $P^{MN} \equiv P^MS_zP^N + P^NS_zP^M$ was introduced. Combining (3.28) with (3.27) one obtains

$$\begin{aligned} & C_s = \frac{-2}{3\pi}\int dp \\ \times & \Big\{F_6^2F_8^2\Big(\left[-3E_1E_3 - 3E_3^2 + 3E_1^2 + 3(E_5+E_7)^2 + E_5^2 + E_7^2 + E_5E_7\right]\frac{\partial_pF_1^2}{(\Delta_{SO}-E_1)^2} + 1 \leftrightarrow 3\Big) \\ - & F_2^2F_4^2\Big(\left[-3E_5E_7 + 3E_5^2 - 3E_7^2 + 3(E_1+E_3)^2 + E_1^2 + E_3^2 + E_1E_3\right]\frac{\partial_pF_5^2}{(\Delta_{SO}-E_5)^2} + 5 \leftrightarrow 7\Big) \\ - & \frac{F_4^2F_6^2}{E_5-E_3}\left((E_3-E_1)^3\frac{\partial_pF_1^2}{(\Delta_{SO}-E_1)^2} - (E_7-E_5)^3\frac{\partial_pF_7^2}{(\Delta_{SO}-E_7)^2}\right) \\ + & \frac{F_2^2F_8^2}{E_7-E_1}\left((E_3-E_1)^3\frac{\partial_pF_3^2}{(\Delta_{SO}-E_3)^2} - (E_7-E_5)^3\frac{\partial_pF_5^2}{(\Delta_{SO}-E_5)^2}\right)\Big\}, \end{aligned} \tag{3.29}$$

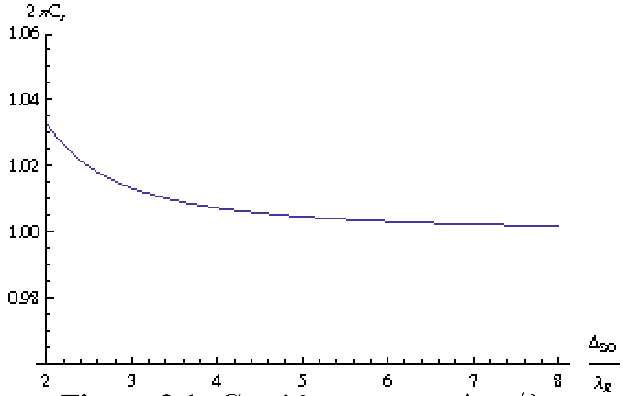

Figure 3.1: C_s with respect to $\Delta_{SO}/\lambda_{\rm R}$.

where $M \leftrightarrow N$ denotes the term which arises from the former entry by interchanging M and N. The detailed calculations are saved for Appendix (A).

The integral in (3.29) can not be managed analytically. However, the numerical calculations are in accord with the result

$$C_s = R(\infty) - R(0), \tag{3.30}$$

where $R(p)$ is deduced to be

$$\begin{aligned} R(p) &= \frac{1}{6\pi}\Big[\frac{\Delta_{SO}-\lambda_{\rm R}}{\sqrt{(\Delta_{SO}-\lambda_{\rm R})^2+p^2}} + \frac{\Delta_{SO}+\lambda_{\rm R}}{\sqrt{(\Delta_{SO}+\lambda_{\rm R})^2+p^2}} \\ &- \frac{\Delta_{SO}}{2\lambda_{\rm R}}\tanh^{-1}\sqrt{1+\frac{p^2}{(\Delta_{SO}-\lambda_{\rm R})^2}} + \frac{\Delta_{SO}}{2\lambda_{\rm R}}\tanh^{-1}\sqrt{1+\frac{p^2}{(\Delta_{SO}+\lambda_{\rm R})^2}}\Big] \end{aligned} \tag{3.31}$$

In the limit $p \to \infty$ (3.31) vanishes, but its $p \to 0$ limit depends on the ratio of the coupling constants Δ_{SO} and λ_R as:

$$\lim_{p\to 0} R(p) = \frac{1}{3\pi}\Big[1 + \frac{1}{4}\frac{\Delta_{SO}}{\lambda_{\rm R}}\ln(\frac{\Delta_{SO}+\lambda_{\rm R}}{\Delta_{SO}-\lambda_{\rm R}})\Big]$$

For diverse values of the coupling constants satisfying $\Delta_{SO} > 2\lambda_{\rm R}$, C_s occurs to be in the range between $0.506/\pi$ and $0.500/\pi$. For example it is found that $C_s = 0.506/\pi$ for $\frac{\Delta_{SO}}{\lambda_{\rm R}} = 3$ and $C_s 0.502/\pi$ for $\frac{\Delta_{SO}}{\lambda_{\rm R}} = 5$. As it is plotted in Figure **3.1**, for $\frac{\Delta_{SO}}{\lambda_{\rm R}} \geq 5$, (3.30) has values closer to $C_s = 1/2\pi$ which is the exact result when $\Delta_{SO} \gg \lambda_R$.

3.4 Discussions

Response of the system (3.4) to the field Ω_μ, which is the spin current j^μ_{spin}, can be obtained from the effective action (3.6) as

$$j^\mu_{spin} = \frac{\delta S_{eff}}{\delta \Omega_\mu}.$$

It is found that the coefficient C_Ω vanishes, so that the spin current provided by the effective action is

$$j^\mu_{spin} = C_s \epsilon^{\mu\nu\rho}\partial_\nu A_\rho,$$

where $C_s \approx e/2\pi$. It is worth mentioning that the spin current is conserved (at least at the tree level), $\partial_\mu j^\mu_{spin} = 0$, though the third component of spin S_z does not commute with the Hamiltonian (3.1). The spatial component of spin current can be interpreted in terms of the spin Hall conductivity σ_{SH} and the electric field $E_a = \partial_a A_0 - \partial_0 A_a$ as

$$j^a_{spin} = \sigma_{SH}\epsilon_{ab}E_a.$$

Thus, one can conclude that for the Kane-Mele model in the presence of Rashba interaction (3.4) it has approximately the quantized value

$$\sigma_{SH} = C_s \approx \frac{1}{2\pi}, \qquad \textbf{(3.32)}$$

for $\Delta_{SO} > 2\lambda_{\mathrm{R}}$. Within the Kane-Mele model in the presence of Rashba interaction, (3.4), a realization of the $2+1$ dimensional spin Hall phase in a certain range of values of the interaction parameters, $\Delta_{SO},\ \lambda_{\mathrm{R}}$, was suggested in [23]. In this molecular graphene construction in order to achieve quantum spin Hall phase, a suitable set of values was given by $\Delta_{SO} = 0.145eV$, and $\lambda_{\mathrm{R}} = 0.04eV$. In fact, adopting these values our numerical calculation produces the result $\sigma_{SH} \approx 1/2\pi$. Moreover, response of the Kane-Mele model (3.4) to the external electromagnetic gauge field A_μ can be derived from the effective action (3.6) as

$$j^\mu_{charge} = \frac{\delta S_{eff}}{\delta A_\mu}.$$

It is shown that the coefficient of the Chern-Simons term, C, vanishes, so that the charge current furnished by the effective action is

$$j^\mu_{charge} \approx \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu \Omega_\rho. \qquad \textbf{(3.33)}$$

In [15] it was demonstrated that (3.33) is the fundamental response equation for the quantum spin Hall effect.

In order to consider the spin Hall phase a topological invariant spin Chern number was introduced in [55]. The definition of [55] relies on the fact that one can project eigenstates of a gapped Hamiltonian to up and down sectors of the spin operator S_z, even if it does not commute with the related Hamiltonian. In [56] this definition was adopted to calculate the spin Chern numbers $N^{sc}_{\pm\uparrow\downarrow}$ for the Kane-Mele model in the presence of Rashba interaction. As it is already noted, $\pm$ and $\uparrow\downarrow$ label the Dirac points and the S_z eigenvalues. They obtained

$$\begin{aligned} N^{sc}_{\pm\uparrow} = K_{\pm\uparrow}(\infty) - K_{\pm\uparrow}(0) &= \frac{1}{2}, \\ N^{sc}_{\pm\downarrow} = K_{\pm\downarrow}(\infty) - K_{\pm\downarrow}(0) &= -\frac{1}{2}, \end{aligned}$$

where

$$\begin{aligned} K_{+\uparrow}(p) &= -K_{+\downarrow}(p) = F_2(p)F_4(p), \\ K_{-\uparrow}(p) &= -K_{-\downarrow}(p) = F_6(p)F_8(p). \end{aligned} \quad \textbf{(3.34)}$$

Now, the “total spin Chern number" relevant to obtain the spin current can be defined as follows:

$$N^{sc} = N^{sc}_{+\uparrow} + N^{sc}_{-\uparrow} - N^{sc}_{-\downarrow} - N^{sc}_{-\downarrow} = 2. \quad \textbf{(3.35)}$$

Although the momentum dependence of $R(p)$ and $K(p)$ given in (3.31) and (3.34) is not the same, the numerical results (3.32) and (3.35) suggest that

$$\sigma_{SH} \approx \frac{1}{4\pi} N^{sc}.$$

Obviously, this suggested relation between the coefficient of effective action C_s, and the “total spin Chern number" N^{sc} needs further clarifications.

It is demonstrated that response of the quantum spin Hall insulator in the presence of Rashba interaction can be obtained from the effective action of the external electromagnetic and spin fields (3.6). Therefore the materials analogous to graphene yield the predictions which are independent of the their detailed structure as far as the underlying Hamiltonian is given by the Kane-Mele model. Their response can be studied within the BF type topological field theory of the external fields.

4. $3+1$ DIMENSIONAL TOPOLOGICAL FIELD THEORY AND THE EFFECTIVE ACTION OF THE $3+1$ DIMENSIONAL TOPOLOGICAL INSULATOR

4.1 Purpose

Topological field theories by definition do not depend on the local features of the space-time in which they are defined. They emerge both in high energy physics [26] and in condensed matter physics. In the latter case they appear as effective theories of quantum matter like topological insulators. Topological insulators are usually defined to be ordinary insulators in the bulk but having conducting edge states on their surface [7]. In the low energy limit, 3+1 dimensional topological insulators subject to electromagnetic fields given by the gauge fields A_μ; $\mu = 0, 1, 2, 3$, are effectively described by the action ($\hbar = c = 1$),

$$S_{3D} = \frac{\theta e^2}{8\pi^2} \int d^4x \, \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma. \tag{4.1}$$

(4.1) is topological in the sense that the metric tensor of the related space-time manifold does not appear in the action and there is no local excitations. Properties of the underlying space-time can be fixed by additional information about the adopted microscopic model. In particle physics (4.1) is known as the topological term for the θ-vacuum [57, 58]. For compact space-time manifolds by choosing $\theta = \pm(2n + 1)\pi$; $n = 0, 1, \cdots$, (4.1) describes the main features of the time reversal invariant topological insulators in $3+1$ dimensions [15].

One of the models which gives rise to the action (4.1) is obtained through a dimensional reduction from the $4+1$ dimensional Dirac theory [15]. One considers the relativistic electrons interacting with the external gauge fields A_μ, as well as with the scalar field $\Theta(x)$ described by

$$\mathcal{L}_{3+1}(\Psi, \bar{\Psi}, A) = \bar{\Psi}[i\gamma^\mu(\partial_\mu - ieA_\mu) + \gamma^4(k_4 + \Theta) - m]\Psi. \tag{4.2}$$

Here, k_4 is a constant and electron charge $e > 0$. γ_4 is one of the gamma matrices introduced in $4+1$ dimensions and $\Theta(x)$ is the reminiscent of the gauge field $A_4(x)$.

Integrating out the fermionic fields $\Psi, \bar{\Psi}$ in the path integral of (4.2), yields the effective action [15, 52]

$$S_A = \frac{e^2}{8\pi^2} \int d^4x \, \Theta(x) \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma. \tag{4.3}$$

$\Theta(x)$ is known in particle physics as axion field (for a review see [27]). For a uniform and constant axion field $\Theta = \theta$, (4.3) yields (4.1).

In this work, a strictly $3 + 1$ dimensional model which leads to (4.1) is proposed. For this aim an action of neutral fermionic quasiparticles will be introduced and it is shown that by integrating out these fermionic fields it is possible to obtain (4.1) through a renormalization procedure. It should be emphasized that these neutral fermions can appear as the quasiparticles of the effective theory. Obviously, it is not possible to construct this neutral spinors by combining the original electrons and holes. It will be shown that this approach can also be generalized to derive the BF type theory which was proposed to describe topological insulators in [59].

4.2 Effective field theory in $3 + 1$ dimensions

Let the field ψ_e denotes the electrons. These electrons interact with external electromagnetic fields A_μ according to the Dirac Lagrangian density

$$\mathcal{L}_D = \bar{\psi}_e \left[i\gamma^\mu (\partial_\mu - ieA_\mu) - m \right] \psi_e. \tag{4.4}$$

On-shell states satisfy the Dirac equation. As it is well known, there are negative energy states of the Dirac equation describing positive charged particles which can be interpreted as antiparticles (holes). Hence, when the excitations of the Dirac particles are considered, one may deal with the particles of both sign. These particles can be tight together to form neutral degrees of freedom. In particular one may assume that in the bulk of a topological insulator effectively there are neutral fermions. This guarantees that there is no electrical transport in the bulk, which is the basic property of an insulator. Obviously, neutral fermions can couple to external electromagnetic fields due to their electric and magnetic dipole moments whose interaction terms are covariantly given by $\sigma^{\mu\nu} F_{\mu\nu}$ and $i\gamma^5 \sigma^{\mu\nu} F_{\mu\nu}$,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu; \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]; \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3.$$

These neutral fermions can be described by the Lagrangian density

$$\mathcal{L}(\bar{\psi}, \psi, F) = \bar{\psi}\left[i\gamma^{\mu}\partial_{\mu} + \frac{1}{2}e\alpha(1 - i\gamma^{5})\sigma^{\mu\nu}F_{\mu\nu} - m\right]\psi, \qquad \textbf{(4.5)}$$

where α is a constant. In the related path integral one can integrate out the fermions to derive the effective action of the electromagnetic field strengths $S[F]$:

$$\exp(iS[F]) \equiv \int D\bar{\psi}D\psi \exp\left(i\int d^{4}x\mathcal{L}(\bar{\psi}, \psi, F)\right).$$

Formally, the effective action can be written as

$$S[F] = -i \ln \det[i\gamma^{\mu}\partial_{\mu} + \frac{1}{2}e\alpha(1 - i\gamma^{5})\sigma^{\mu\nu}F_{\mu\nu} - m]. \qquad \textbf{(4.6)}$$

As usual, it is preferable to work in the momentum space. One of the terms which (4.6) generates is

$$\int \frac{d^{4}k_{1}}{(2\pi)^{4}}\frac{d^{4}k_{2}}{(2\pi)^{4}}F_{\mu\nu}(k_{1})F_{\rho\sigma}(k_{2})\pi^{\mu\nu\rho\sigma}(k_{1})\delta^{4}(k_{1} - k_{2}). \qquad \textbf{(4.7)}$$

In the low energy limit where momentum of the external legs of the related Feynman diagram vanishes (4.7) can be taken as the effective action. At the one loop level $\pi^{\mu\nu\rho\sigma}(k)$ can be written as

$$\pi^{\mu\nu\rho\sigma}(k) = \frac{i}{2}e^{2}\int \frac{d^{4}p}{(2\pi)^{4}}\mathrm{tr}[(\frac{\alpha}{2}(1-i\gamma^{5})\sigma^{\mu\nu}G_{F}(p))(\frac{\alpha}{2}(1-i\gamma^{5})\sigma^{\rho\sigma}G_{F}(p-k))], \qquad \textbf{(4.8)}$$

where the Feynman propagator $G_F(p)$ is defined as

$$G_{F}(p) = \frac{i(\gamma^{\mu}p_{\mu} + m)}{p^{2} - m^{2}}.$$

Using the properties of Dirac matrices in $3 + 1$ dimensions, one can show that (4.8) reduces to two terms:

$$\pi^{\mu\nu\rho\sigma}(k) = \Sigma^{\mu\nu\rho\sigma}(k) + \Pi^{\mu\nu\rho\sigma}(k). \qquad \textbf{(4.9)}$$

They are defined as

$$\begin{aligned}
\Sigma^{\mu\nu\rho\sigma}(k) &= \frac{i}{16}e^{2}\alpha^{2}\int \frac{d^{4}p}{(2\pi)^{4}}\frac{p_{\alpha}(p - k)_{\beta}\mathrm{tr}\{[\gamma^{\mu}, \gamma^{\nu}]\gamma^{\alpha}[\gamma^{\rho}, \gamma^{\sigma}]\gamma^{\beta}}{(p^{2} - m^{2})((p - k)^{2} - m^{2})}, \\
\Pi^{\mu\nu\rho\sigma}(k) &= \frac{1}{16}e^{2}\alpha^{2}m^{2}\int \frac{d^{4}p}{(2\pi)^{4}}\frac{\mathrm{tr}\{[\gamma^{\mu}, \gamma^{\nu}][\gamma^{\rho}, \gamma^{\sigma}]\gamma^{5}\}}{(p^{2} - m^{2})((p - k)^{2} - m^{2})}.
\end{aligned}$$

$\Sigma^{\mu\nu\rho\sigma}(k)$ is quadratically divergent. In the low energy limit $(k \to 0)$ it leads to

$$\int \frac{d^{4}p}{(2\pi)^{4}}\frac{p_{\alpha}p_{\beta}}{(p^{2} - m^{2})^{2}} = \left[\frac{\Lambda^{2}}{16\pi^{2}} + \frac{m^{2}}{4\pi^{2}}\ln\left(\frac{\Lambda}{m}\right) + \cdots\right]\eta_{\alpha\beta},$$

where $\eta_{\alpha\beta}$ is the metric tensor and Λ is the ultraviolet cut-off. Although this term is divergent for $\Lambda \to \infty$, it is multiplied with the trace term which can be seen to vanish in 4 dimensions:

$$\mathrm{tr}\{[\gamma^\mu, \gamma^\nu]\gamma^\alpha[\gamma^\rho, \gamma^\sigma]\gamma_\alpha\} = 0.$$

Thus one concludes

$$\Sigma^{\mu\nu\rho\sigma}(0) = 0.$$

The other term $\Pi^{\mu\nu\rho\sigma}$ in (4.9) is topological, i.e. it is independent of the space-time metric $\eta_{\mu\nu}$. In terms of the totally antisymmetric Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$, it is originated from the following trace of the γ matrices:

$$\mathrm{tr}\left\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right\} = -4i\epsilon^{\mu\nu\rho\sigma}.$$

One can express $\Pi^{\mu\nu\rho\sigma}(k)$ as

$$\Pi^{\mu\nu\rho\sigma}(k) = e^2m^2\alpha^2\epsilon^{\mu\nu\rho\sigma}I(k),$$

where

$$I(k) = -i\int \frac{d^4p}{(2\pi)^4}\frac{1}{((p^2-m^2)((p-k)^2-m^2)}, \tag{4.10}$$

whose integrand is spherically symmetric in the low energy limit $(k \to 0)$. The integral is logarithmically divergent but can be dimensionally regularized. In order to calculate (4.10) in d dimensions we introduce the parameter μ having the dimension of mass such that the integral

$$I(0) = -i\mu^{4-d}\int \frac{d^dp}{(2\pi)^4}\frac{1}{(p^2-m^2)^2}, \tag{4.11}$$

remains dimensionless. Calculation of $I(0)$ in d dimensions leads to [60]

$$I(0) = \frac{1}{4\pi^{d/2}}\Gamma(2-d/2)(\frac{\mu^2}{m^2})^{2-d/2},$$

where $\Gamma(x)$ is the gamma function.

Setting $d = 4 - \varepsilon$ the finite and infinite parts in the $\epsilon \to 0$ limit are separated as

$$I(0) = \frac{1}{(4\pi)^2}(\frac{2}{\epsilon} + \ln(\frac{\mu^2}{m^2}) + \ln(4\pi) - \gamma + ...).$$

γ is the Euler-Mascheroni constant. The bare coupling constant α, or the mass m, can be renormalized with the divergent constant Z to introduce the finite parameter θ_F as

$$Zm^2\alpha^2I(0) = \frac{\theta_F}{32\pi^2}.$$

As it is usual in a renormalized theory, the value of θ_F will be fixed in terms of the "experimentally measured quantities." One concludes that

$$\pi^{\mu\nu\rho\sigma}(0) = \frac{e^2\theta_F}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}.$$

Therefore, in the low energy limit the renormalized effective action is

$$S[F] = \frac{e^2\theta_F}{32\pi^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(x)F_{\rho\sigma}(x). \qquad \textbf{(4.12)}$$

It is the same with the action (4.1) for $\theta_F = \theta$.

Properties of the space-time on which the theory is defined should be dictated by the physical arguments. It can be assumed that the neutral quasiparticles of the initial Lagrangian density (4.5) is composed of the original particles and holes. For example in the case of topological insulators the physical particles are the electrons minimally coupled to the gauge fields A_μ as in (4.4). Then, there are some configurations which are consistent with periodic space-time yielding the quantization condition

$$\frac{e^2}{32\pi^2}\int d^4x\ \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = N, \qquad \textbf{(4.13)}$$

where N is an integer. Thus the partition functions for $\theta_F = \pm(2n{+}1)\pi;\ n = 0, 1, \cdots$, are the same, so that the theory is time reversal invariant. Obviously, there is no a priori given condition for the space-time structure of the topological field theory action (4.12). One may choose the space-time to be periodic but yielding the quantization condition

$$\frac{e^2}{32\pi^2}\int d^4x\ \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = N_f^2 N, \qquad \textbf{(4.14)}$$

where N_f is an odd integer. Then, set $\theta_F = \pm(2n+1)\pi/N_f;\ n = 0, 1, \cdots$, which defines the time reversal invariant fractional topological insulator [28–30]. In this case, as in [28], the partons denoted ψ_p, whose electromagnetic interaction Lagrangian density is given by

$$\mathcal{L}_p = \bar{\psi}_p\left[i\gamma^\mu(\partial_\mu - i\frac{e}{N_f}A_\mu) - m\right]\psi_p,$$

can be chosen as the physical particles in the bulk of the fractional topological insulator. They possess the electric charge e/N_f, so that the quantization condition (4.14) is justified.

4.3 BF theory

It was argued in [59] that the time reversal invariant $3+1$ dimensional topological insulator is described by the BF type effective field theory given by

$$\mathcal{L}_{BF} = \frac{e}{2\pi}\epsilon^{\mu\nu\rho\sigma} a_\mu \partial_\nu b_{\rho\sigma} + \frac{e}{2\pi}\epsilon^{\mu\nu\rho\sigma} A_\mu \partial_\nu b_{\rho\sigma} + C\epsilon^{\mu\nu\rho\sigma}\partial_\mu a_\nu \partial_\rho A_\sigma, \quad \textbf{(4.15)}$$

where C is a constant parameter. To fix the value of it as $C = \pm e^2/8\pi$, the gauge charge lattice was specified as an additional information. Thus, integrating out a_μ and $b_{\rho\sigma}$ fields in the related path integral yields

$$S_{3D} = \pm\frac{e^2}{8\pi}\int d^4x \epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma, \quad \textbf{(4.16)}$$

which is the same with (4.1) for $\theta = \pm 1$. It is possible to construct (4.15) in terms of the procedure outlined in Section 4.2. For this purpose one may introduce the neutral fermions denoted $\psi_A;\ A = 1, 2, \cdots, 6$ and define the Lagrangian density as

$$\begin{aligned}
\mathcal{L} &= \bar{\psi}_1\left[i\gamma^\mu\partial_\mu + \frac{1}{2}\lambda(1-i\gamma^5)\sigma^{\mu\nu}(F_{\mu\nu}+b_{\mu\nu}) - m\right]\psi_1 \\
&+ \bar{\psi}_2\left[i\gamma^\mu\partial_\mu + \frac{1}{2}\lambda(1+i\gamma^5)\sigma^{\mu\nu}(F_{\mu\nu}-b_{\mu\nu}) - m\right]\psi_2 \\
&+ \bar{\psi}_3\left[i\gamma^\mu\partial_\mu + \frac{1}{4}\beta(1-i\gamma^5)\sigma^{\mu\nu}(F_{\mu\nu}+f_{\mu\nu}) - m\right]\psi_3 \\
&+ \bar{\psi}_4\left[i\gamma^\mu\partial_\mu + \frac{1}{4}\beta(1+i\gamma^5)\sigma^{\mu\nu}(F_{\mu\nu}-f_{\mu\nu}) - m\right]\psi_4 \\
&+ \bar{\psi}_5\left[i\gamma^\mu\partial_\mu + \frac{1}{2}\lambda(1-i\gamma^5)\sigma^{\mu\nu}(f_{\mu\nu}+b_{\mu\nu}) - m\right]\psi_5 \\
&+ \bar{\psi}_6\left[i\gamma^\mu\partial_\mu + \frac{1}{2}\lambda(1+i\gamma^5)\sigma^{\mu\nu}(f_{\mu\nu}-b_{\mu\nu}) - m\right]\psi_6,
\end{aligned}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. Now, it is possible to integrate out each $\psi_A, \bar{\psi}_A$ neutral fermion degrees of freedom following the procedure of Section 4.2. Obviously, at the one loop level integrals are divergent which can be expressed in the low energy limit as $I(0)$ which is defined in (4.11). One may renormalize the bare coupling constants λ and β to define the finite constants Λ_F and C_F as

$$Zm^2\lambda^2 I(0) = \Lambda_F, \ \ Zm^2\beta^2 I(0) = C_F.$$

Then, the low energy effective action can be written as

$$S[A,b,a] = \int d^4x \left\{\Lambda_F\left[\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu b_{\rho\sigma} + \epsilon^{\mu\nu\rho\sigma}\partial_\mu a_\nu b_{\rho\sigma}\right] + C_F\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho a_\sigma\right\}. \quad \textbf{(4.17)}$$

As usual the renormalized quantities should be fixed by some additional conditions. One can choose $\Lambda_F = 1/2\pi$ and $C_F = \pm e^2/8\pi$, so that (4.17) yields the BF theory described by (4.15), up to surface terms. Apparently, integrating out first $b_{\mu\nu}$ then a_μ fields yields the action (4.16).

4.4 Discussions

By integrating out the neutral Dirac particles described by the Lagrangian density (4.5), the effective action (4.12) is constructed in the low energy limit. The procedure of calculating the effective topological field theory (4.12) does not refer to the origin of this neutral fermions. They are supposed to be the quasiparticle excitations of the effective theory. However, neutral fermion excitations have already been appeared in the study of the Hall effect for even filling fractions [61, 62]. Moreover, neutral Dirac fermions composed of two Majorana fermions arise in systems composed of topological, magnetic and superconducting insulators [63]. The original physical particles establish the features of the space-time manifold on which the effective topological field theory is defined. Then, according to the quantization conditions like (4.13) and (4.14), the value of the physical coupling constant θ_F should be chosen.

Interchanging Abelian field strengths with non-Abelian ones in the Lagrangian density (4.5) will not alter the construction of the low energy effective action (4.12) presented in Section 4.2. Therefore, this procedure is also valid for non-Abelian gauge fields. In this case, quantization condition of the effective action will be changed [64], so that the renormalized coefficient θ_F should be appropriately chosen.

5. SEMICLASSICAL CALCULATION OF CHIRAL MAGNETIC EFFECT AND CHIRAL ANOMALY IN $d + 1$ DIMENSIONS

In this part of the thesis, $d + 1$ dimensional Weyl Hamiltonian which is derived from the massless Dirac Hamiltonian is considered. Through the diagonalization procedure of the Weyl Hamiltonian, a matrix valued Berry gauge field can be extracted. By means of the Berry field strength, a matrix valued symplectic two-form is introduced. Phase space volume form is defined appropriately. The chiral current due to the chiral Weyl particles interacting with the electromagnetic fields is constructed. A semiclassical formulation of both CME and chiral anomaly is provided for all even spacetime dimensions.

5.1 Purpose

Semiclassical analysis of dynamical systems is useful to get a better understanding of some quantum mechanical phenomena especially in many body systems. The recent studies [31, 32] show that it is possible to embody chiral anomaly and chiral magnetic effect within the semiclassical chiral kinetic theory. In order to achieve this [31] deformed the initial phase space with a monopole field emerging from the Berry curvature [33]. In [32] it is denoted that this monopole field is the the field strength of the Berry gauge field which is constructed from the diagonalization of the $3 + 1$ dimensional Weyl Hamiltonian. A formulation of anomalies in higher dimensional spacetimes within the classical chiral kinetic theory was carried out in [34]. However their method is unclear to construct explicit form of the equations of motion which would yield a generalization of the CME in higher dimensions. On the other hand CME for even $d + 1$ dimensions was only conjectured in [35]. Having inspired from the approach of [36], the aim of this part is due to a formulation of the chiral magnetic effect and chiral anomaly for all even $d + 1$ dimensions within the same semiclassical chiral kinetic theory.

5.2 Weyl Hamiltonian and the Berry Gauge Field

The massless Dirac Hamiltonian in $d+1$ dimensions is denoted as

$$\mathcal{H} = \boldsymbol{\alpha} \cdot \boldsymbol{p}, \tag{5.1}$$

in $\hbar = c = 1$, units. $\boldsymbol{p}$ is the d dimensional momentum vector and $\alpha_{\rm A};\ {\rm A} = 1, ..., d$, are the $2^{[(d+1)/2]} \times 2^{[(d+1)/2]}$ dimensional matrices satisfying the anticommutation relations

$$\{\alpha_{\rm A}, \alpha_{\rm B}\} = 2\delta_{\rm AB}. \tag{5.2}$$

They can be expressed as $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$, in terms of the gamma matrices which obey the Clifford algebra, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, where the metric tensor is $g^{\mu\nu} = \mathrm{diag}(1, -1, \cdots, -1);\ \mu, \nu = 0, 1, ..., d$. In the chiral representation $\boldsymbol{\alpha}$ is block diagonal, so that (5.1) yields the Weyl Hamiltonian

$$\mathcal{H}_{\rm W} = \boldsymbol{\Sigma} \cdot \boldsymbol{p}, \tag{5.3}$$

where $\Sigma_{\rm A}$ are $2^{[(d-1)/2]} \times 2^{[(d-1)/2]}$ matrices. Quantum mechanical description of the Weyl particle is furnished with the Weyl equation

$$\mathcal{H}_{\rm W} \psi_E(\boldsymbol{p}) = E \psi_E(\boldsymbol{p}).$$

The energy eigenvalues are $E = (p, -p)$ where $p = |\boldsymbol{p}|$. Focusing on the positive energy solutions $|\psi^\alpha\rangle;\ \alpha = 1, ..., 2^{[\frac{d-3}{2}]}$, one can define the Berry gauge field:

$$\mathcal{A}_{\rm A}^{\alpha\beta} = i\langle \psi^\alpha | \frac{\partial}{\partial p_{\rm A}} | \psi^\beta \rangle. \tag{5.4}$$

$\mathcal{A}$ is Abelian for $d = 3$, however it becomes to be non-Abelian for higher dimensions. Thus, in general the Berry field strength is given by

$$\mathcal{G}_{\rm AB}^{\alpha\beta} = \frac{\partial \mathcal{A}_{\rm B}^{\alpha\beta}}{\partial p_{\rm A}} - \frac{\partial \mathcal{A}_{\rm A}^{\alpha\beta}}{\partial p_{\rm B}} - i[\mathcal{A}_{\rm A}, \mathcal{A}_{\rm B}]^{\alpha\beta}. \tag{5.5}$$

In the following parts $d = 3$ and $d = 5$ dimensional Berry gauge fields will be presented explicitly.

5.3 Chiral Anomaly and Chiral Magnetic Effect in $3+1$ Dimensions

In [32] it was observed that for $d = 3$, the Berry field strength extracted from the diagonalization of Weyl Hamiltonian describes a monopole located at $\boldsymbol{p} = 0$. They clarified the role of this monopole field in obtaining the CME and the chiral anomaly. In [34] it was argued that in all even dimensions chiral (non-Abelian) anomaly arises in the presence of the Berry gauge fields within the differential form formulation of classical Hamiltonian dynamics. Here, the approach of [34] is reviewed basically considering external electromagnetic fields in $d = 3$. However, it will be also shown how this formalism can be employed to acquire the first time derivatives of phase space variables which are needed to formulate the CME as in [32].

In the chiral representation of gamma matrices

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \textbf{(5.6)}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli spin matrices, the Weyl Hamiltonian is acquired as

$$\mathcal{H}_{\mathrm{W}}{}^{(3)} = \boldsymbol{\sigma} \cdot \boldsymbol{p} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}. \qquad \textbf{(5.7)}$$

To formulate the Hamiltonian dynamics one introduces the one-form

$$\eta_H = p_a dx_a + A_a(x,t)dx_a - \mathcal{A}_a(p)dp_a - Hdt,$$

where $a = 1, 2, 3$. $\boldsymbol{A}$ is the vector potential of the external magnetic field $\boldsymbol{B}$:

$$F_{ab} = \frac{\partial A_b}{\partial x_a} - \frac{\partial A_a}{\partial x_b} = \epsilon_{abc} B_c.$$

$H = p + A_0$ where p is the Weyl Hamiltonian (5.7) diagonalized and projected onto the positive energy eigenstate. $A_0(x,t)$ is the scalar potential of the external electric field $\boldsymbol{E} = \partial \boldsymbol{A}/\partial t - \boldsymbol{\nabla} A_0$. $\mathcal{A}_a(p)$ is the Abelian Berry gauge field. Inspecting the positive energy solution of (5.7) one can show that its field strength is given in terms of $\boldsymbol{b} = \hat{\boldsymbol{p}}/2p^2$, as

$$G_{ab} = \frac{\partial \mathcal{A}_b}{\partial p_a} - \frac{\partial \mathcal{A}_a}{\partial p_b} = \epsilon_{abc} b_c. \qquad \textbf{(5.8)}$$

It is the field of a monopole located at $\boldsymbol{p} = 0 : \boldsymbol{\nabla} \cdot \boldsymbol{b} = 2\pi\delta^3(p)$.

The exterior derivative of η_H provides the symplectic two-form

$$w_H = d\eta_H = dp_a \wedge dx_a + F - G - \hat{p}_a dp_a \wedge dt + E_a dx_a \wedge dt,$$

where $F = \frac{1}{2}F_{ab}\, dx_a \wedge dx_b$ and $G = \frac{1}{2}G_{ab}\, dp_a \wedge dp_b$. In terms of w_H the volume form in $6+1$ dimensional phase space is defined by

$$\Omega \equiv \frac{1}{3!} w_H^3 \wedge dt. \tag{5.9}$$

It can be also written in terms of the canonical volume element of the phase space $dV^{(3)}$:

$$\Omega = \sqrt{w}\, dV^{(3)} \wedge dt. \tag{5.10}$$

Here, $\sqrt{w} \equiv \sqrt{\det(w)}$, is the Pfaffian of the matrix

$$\begin{pmatrix} F_{ab} & -\delta_{ab} \\ \delta_{ab} & -G_{ab} \end{pmatrix}.$$

It was calculated explicitly in [33] as $\sqrt{w} = (1 + \boldsymbol{B} \cdot \boldsymbol{b})$.

The equations of motion can be derived by demanding that w_H satisfies

$$i_v w_H = 0, \tag{5.11}$$

which is the interior product of the vector field

$$v = \frac{\partial}{\partial t} + \dot{x}_a \frac{\partial}{\partial x_a} + \dot{p}_a \frac{\partial}{\partial p_a},$$

with w_H. In fact, (5.11) gives rise to the equations of motion obtained in [32]:

$$\dot{p}_a = \dot{x}_b F_{ab} + E_a, \tag{5.12a}$$

$$\dot{x}_a = \dot{p}_b G_{ab} + \hat{p}_a. \tag{5.12b}$$

Liouville equation will be obtained by the time evolution of the volume form Ω which can be found by calculating its Lie derivative associated with v. It can be accomplished in two different ways. First one can employ (5.10) to observe that

$$L_v\Omega = (i_v d + d i_v)\left(\sqrt{w} dV^{(3)} \wedge dt\right) = \left(\frac{\partial}{\partial t}\sqrt{w} + \frac{\partial}{\partial x_a}(\sqrt{w}\dot{x}_a) + \frac{\partial}{\partial p_a}(\sqrt{w}\dot{p}_a)\right) dV^{(3)} \wedge dt. \tag{5.13}$$

Then, it is possible to show that the definition (5.9) implies

$$L_v\Omega = (i_v d + d i_v)\left(\frac{1}{3!} w_H^3 \wedge dt\right) = d i_v \left(\frac{1}{3!} w_H^3 \wedge dt\right) = \frac{1}{2} dw_H \wedge w_H^2,$$

where

$$dw_{_H} = -\frac{1}{2}\frac{\partial G_{ab}}{\partial p_c} dp_c \wedge dp_a \wedge dp_b.$$

Making use of (5.8) one can express $dw_{_H}$ as

$$dw_{_H} = -(\boldsymbol{\nabla} \cdot \boldsymbol{b})\, dp_1 \wedge dp_2 \wedge dp_3 = -2\pi\delta^3(p)\, dp_1 \wedge dp_2 \wedge dp_3.$$

The unique contribution from $w_{_H}^2$ will be $E_a F_{bc} dx_a \wedge dx_b \wedge dx_c \wedge dt$, thus one finds that

$$\frac{1}{2} dw_{_H} \wedge w_{_H}^2 = 2\pi\delta^3(p)(\boldsymbol{E} \cdot \boldsymbol{B})\, dV^{(3)} \wedge dt.$$

So that the Liouville equation is anomalous:

$$\left(\frac{\partial}{\partial t}\sqrt{w} + \frac{\partial}{\partial x_a}(\sqrt{w}\dot{x}_a) + \frac{\partial}{\partial p_a}(\sqrt{w}\dot{p}_a)\right) = 2\pi\delta^3(p)\boldsymbol{E} \cdot \boldsymbol{B}. \tag{5.14}$$

In quantum field theory the chiral anomaly is expressed as non-conservation of the classically conserved chiral current at the quantum level. In order to connect the anomaly contribution in (5.14) to the non-conservation of particle number introduce the probability function $f(x, p, t)$, which satisfies the collisionless Boltzmann equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_a}\dot{x}_a + \frac{\partial f}{\partial p_a}\dot{p}_a = 0. \tag{5.15}$$

It is appropriate to define the probability density function as $\rho(x, p, t) = \sqrt{w} f$. Hence, defining the particle number density and the particle current density as

$$n(x, t) = \int \frac{d^3p}{(2\pi)^3}\rho(x, p, t), \quad j_a = \int \frac{d^3p}{(2\pi)^3}\rho(x, p, t)\dot{x}_a.$$

and utilizing (5.14) and (5.15) one can observe that

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\dot{x}_a)}{\partial x_a} + \frac{\partial(\rho\dot{p}_a)}{\partial p_a} = 2\pi f \delta^3(p)\boldsymbol{E} \cdot \boldsymbol{B}. \tag{5.16}$$

In order to derive non-conservation of the particle current one can integrate (5.16) over the momentum degrees of freedom. There is no influx of particles from the negative energy sea because only the positive energy sector of the Weyl Hamiltonian is taken into account. Thus, the momentum current density $\boldsymbol{j}_p = \rho\dot{\boldsymbol{p}}$ vanishes at the boundary of the momentum space:

$$\int \frac{d^3p}{(2\pi)^3}\frac{\partial(\rho\dot{p}_a)}{\partial p_a} = 0.$$

Actually, the Berry monopole situated on the boundary $|\boldsymbol{p}| = 0$ is responsible for the non-conservation of the chiral particle current. Therefore, the statement of non-conservation of the particle current follows,

$$\frac{\partial n(x,t)}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = \frac{1}{4\pi^2} f(x, p=0, t) \boldsymbol{E} \cdot \boldsymbol{B}.$$

To derive the CME in line with [32] one needs to solve the equations of motion (5.12) for $\sqrt{w}\dot{x}_a$. Obviously, here this can be done directly, because $\sqrt{w}$ has already been calculated in [33]. However, it is possible to obtain the same result by inspecting the explicit form of w_H^3. As a byproduct the explicit form of $\sqrt{w}$ will also be provided. To present this method which will be extremely useful in higher dimensions, one writes w_H^3 explicitly as

$$\begin{aligned} w_H^3 &= dp_a \wedge dx_a \wedge dp_b \wedge dx_b \wedge dp_c \wedge dx_c - 6F \wedge \mathcal{G} \wedge dp_a \wedge dx_a \\ &\quad -3\hat{p}_a dp_a \wedge dt \wedge dp_b \wedge dx_b \wedge dp_c \wedge dx_c - 6E_a F \wedge \mathcal{G} \wedge dx_a \wedge dt \\ &\quad +3E_a dx_a \wedge dt \wedge dp_b \wedge dx_b \wedge dp_c \wedge dx_c - 6\hat{p}_a F \wedge dp_a \wedge dt \wedge dp_b \wedge dx_b \\ &\quad -6E_a \mathcal{G} \wedge dx_a \wedge dt \wedge dp_b \wedge dx_b + 6\hat{p}_a F \wedge \mathcal{G} \wedge dp_a \wedge dt. \end{aligned}$$

Its Lie derivative associated with v procures

$$\begin{aligned} L_v \Omega = \frac{1}{3!} dw_H^3 &= \Big\{ \frac{\partial}{\partial t}(1 + \boldsymbol{B} \cdot \boldsymbol{b}) + \frac{\partial}{\partial \boldsymbol{x}} \left(\hat{\boldsymbol{p}} + \boldsymbol{E} \times \boldsymbol{b} + \boldsymbol{B}(\hat{\boldsymbol{p}} \cdot \boldsymbol{b})\right) \\ &\quad + \frac{\partial}{\partial \boldsymbol{p}} \left(\boldsymbol{E} + \hat{\boldsymbol{p}} \times \boldsymbol{B} + \boldsymbol{b}(\boldsymbol{E} \cdot \boldsymbol{B})\right) \Big\} dV^{(3)} \wedge dt. \end{aligned} \tag{5.17}$$

Comparing (5.17) with (5.13) one can directly read $\sqrt{w}$, as well as the solutions of the equation of motions (5.12) as

$$\begin{aligned} \sqrt{w} &= 1 + \boldsymbol{B} \cdot \boldsymbol{b}, \\ \sqrt{w}\dot{\boldsymbol{x}} &= \hat{\boldsymbol{p}} + \boldsymbol{E} \times \boldsymbol{b} + \boldsymbol{B}(\hat{\boldsymbol{p}} \cdot \boldsymbol{b}), \\ \sqrt{w}\dot{\boldsymbol{p}} &= \boldsymbol{E} + \hat{\boldsymbol{p}} \times \boldsymbol{B} + \boldsymbol{b}(\boldsymbol{E} \cdot \boldsymbol{B}). \end{aligned}$$

Now, the particle current density $\boldsymbol{j}$ can be obtained as

$$\boldsymbol{j} = \int \frac{d^3p}{(2\pi)^3} \sqrt{w} \dot{\boldsymbol{x}} f = \int \frac{d^3p}{(2\pi)^3} \hat{\boldsymbol{p}} f + \boldsymbol{E} \times \int \frac{d^3p}{(2\pi)^3} \boldsymbol{b} f + \boldsymbol{B} \int \frac{d^3p}{(2\pi)^3} \hat{\boldsymbol{p}} \cdot \boldsymbol{b} f.$$

The last term where the current is parallel to the magnetic field is the CME term that was mentioned in [32].

5.4 Chiral Anomaly and Chiral Magnetic Effect in $5+1$ Dimensions

5.4.1 $5+1$ dimensional Berry gauge field

In $5+1$ dimensions a representation of $\alpha_i;\ i=1,..,5$, which satisfy (5.2) can be given as the direct product of the $3+1$ dimensional gamma matrices (5.6) with $\sigma_0 = \mathrm{diag}(1,1)$ and $\sigma_3 = \mathrm{diag}(1,-1)$:

$$\alpha_1 = \sigma_0 \otimes \gamma_0\gamma_1, \quad \alpha_2 = \sigma_0 \otimes \gamma_0\gamma_2, \quad \alpha_3 = \sigma_0 \otimes \gamma_0\gamma_3, \quad \alpha_4 = i\sigma_0 \otimes \gamma_0\gamma_5, \quad \alpha_5 = \sigma_3 \otimes \gamma_0.$$

Using this representation one can write the massless Dirac Hamiltonian in block diagonal form in terms of the Weyl Hamiltonian (5.3), where the $\mathbf{\Sigma}$ matrices are

$$\Sigma_a = \begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix};\ {\scriptstyle a=1,2,3,} \quad \Sigma_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \Sigma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Therefore, the Weyl Hamiltonian in $5+1$ dimensions is expressed as

$$\mathcal{H}_{\mathrm{W}} = \begin{pmatrix} \sigma_a p_a & i(p_4 + ip_5) \\ -i(p_4 - ip_5) & -\sigma_a p_a \end{pmatrix}.$$

The normalized, positive energy eigenstates $\mathcal{H}_{\mathrm{W}}|\psi^\alpha\rangle = p|\psi^\alpha\rangle;\ \alpha = 1,2$, are found to be

$$|\psi^1\rangle = \sqrt{\frac{p_4^2+p_5^2}{2p(p-p_3)}} \begin{pmatrix} \frac{i(p_1-ip_2)}{p_4-ip_5} \\ \frac{i(p-p_3)}{p_4-ip_5} \\ 0 \\ 1 \end{pmatrix}, \quad |\psi^2\rangle = \frac{1}{\sqrt{2p(p-p_3)}} \begin{pmatrix} i(p_4+ip_5) \\ 0 \\ p-p_3 \\ -(p_1+ip_2) \end{pmatrix}.$$

One substitutes these degenerate eigenvectors into the definition (5.4) and build the non-Abelian Berry gauge field components $\mathcal{A}_i^{\alpha\beta}$ as

$$\begin{aligned}
\mathcal{A}_1 &= \frac{1}{2p(p-p_3)} \begin{pmatrix} -p_2 & -i\sqrt{p_4^2+p_5^2} \\ i\sqrt{p_4^2+p_5^2} & p_2 \end{pmatrix}, \\
\mathcal{A}_2 &= \frac{1}{2p(p-p_3)} \begin{pmatrix} p_1 & \sqrt{p_4^2+p_5^2} \\ \sqrt{p_4^2+p_5^2} & -p_1 \end{pmatrix}, \\
\mathcal{A}_3 &= 0, \\
\mathcal{A}_4 &= \frac{1}{2p(p-p_3)} \begin{pmatrix} p_5[\frac{2p(p-p_3)}{p_4^2+p_5^2} - 1] & \frac{i\sqrt{p_4^2+p_5^2}(p_1+ip_2)}{p_4+ip_5} \\ \frac{-i\sqrt{p_4^2+p_5^2}(p_1-ip_2)}{p_4-ip_5} & p_5 \end{pmatrix}, \\
\mathcal{A}_5 &= \frac{1}{2p(p-p_3)} \begin{pmatrix} -p_4[\frac{2p(p-p_3)}{p_4^2+p_5^2} - 1] & \frac{-\sqrt{p_4^2+p_5^2}(p_1+ip_2)}{p_4+ip_5} \\ \frac{-\sqrt{p_4^2+p_5^2}(p_1-ip_2)}{p_4-ip_5} & -p_4 \end{pmatrix}.
\end{aligned} \tag{5.18}$$

By employing (5.18) in the definition (5.5) it is possible to extract $\mathcal{G}_{ij}^{\alpha\beta}$ as presented in Appendix B.

5.4.2 Liouville equation and the chiral anomaly in $5+1$ dimensions

One would like to consider kinetic theory of the $5+1$ dimensional Weyl particles in the presence of the Berry gauge field (5.18) and the external electromagnetic fields $\mathcal{F}_{ij} = \partial A_j/\partial x_i - \partial A_i/\partial x_j$, $\mathcal{E}_i = -\partial A_0/\partial x_i + \partial A_i/\partial t$, generated by the scalar and vector potentials $A_0(x,t)$ and $\boldsymbol{A}(x,t)$. In order to acquire the chiral anomaly one can mainly follow the procedure of Section 5.3 with some modifications required to deal with spin degrees of freedom. Dealing with the non-Abelian Berry gauge fields $\mathcal{A}_i$, the appropriate definition of the matrix valued symplectic two-form is

$$\tilde{w}_H \equiv dp_i \wedge dx_i + \mathcal{F} - \mathcal{G} - \hat{p}_i dp_i \wedge dt + \mathcal{E}_i dx_i \wedge dt,$$

where $\mathcal{G} = \frac{1}{2}\mathcal{G}_{ij}\; dp_i \wedge dp_j$ and $\mathcal{F} = \frac{1}{2}\mathcal{F}_{ij}\; dx_i \wedge dx_j$. Here, the matrix indices are suppressed and the related 2×2 unit matrix I was not written explicitly.

One defines the matrix valued vector field

$$\tilde{v} = \frac{\partial}{\partial t} + \dot{X}_i \frac{\partial}{\partial x_i} + \dot{P}_i \frac{\partial}{\partial p_i}, \qquad \textbf{(5.19)}$$

introducing the matrix valued time evolutions of the phase space variables:

$$\dot{X}_i = \begin{pmatrix} \dot{X}_i^{11} & \dot{X}_i^{12} \\ \dot{X}_i^{21} & \dot{X}_i^{22} \end{pmatrix}, \quad \dot{P}_i = \begin{pmatrix} \dot{P}_i^{11} & \dot{P}_i^{12} \\ \dot{P}_i^{21} & \dot{P}_i^{22} \end{pmatrix}.$$

It is a good point to remind that also in the canonical formulation of the Dirac particle one obtains matrix valued velocity which is similar to the above expression though the phase space variables are not matrix.

Acquiring the equations of motion in terms of the interior product of (5.19) with $\tilde{w}_H$ as

$$i_{\tilde{v}} \tilde{w}_H = 0. \qquad \textbf{(5.20)}$$

(5.20) yields the following equations of motion which are plausible,

$$\dot{P}_i = \dot{X}_j \mathcal{F}_{ij} + \mathcal{E}_i,$$
$$\dot{X}_i = \mathcal{G}_{ij} \dot{P}_j + \hat{p}_i.$$

The novelty in this formulation is the fact the spin degrees of freedom are not treated as dynamical variables. Classical dynamics is asserted through the phase space variables $(\boldsymbol{x}, \boldsymbol{p})$, so that the volume form is defined as

$$\tilde{\Omega} \equiv -\frac{1}{5!} \tilde{w}_H^5 \wedge dt. \qquad \textbf{(5.22)}$$

It can be expressed as

$$\tilde{\Omega} = \tilde{w}_{1/2}\, dV^{(5)} \wedge dt, \tag{5.23}$$

where $dV^{(5)}$ is the canonical volume element (Liouville measure) of the 10 dimensional phase space (x_i, p_i). $\tilde{w}_H$ is a two-form in the phase space variables (x_i, p_i), so that $\tilde{w}_{1/2}$ is the Pfaffian of the 10×10 matrix

$$\begin{pmatrix} \mathcal{F}_{ij} & -\delta_{ij} \\ \delta_{ij} & -\mathcal{G}_{ij} \end{pmatrix}.$$

The explicit form of $\tilde{w}_{1/2}$ will be provided in Section 5.4.3.

One would like to obtain Liouville equation so that she/he considers the Lie derivative of $\tilde{\Omega}$ associated with the vector field (5.19). It can be expressed either using the definition (5.23) as

$$L_{\tilde{v}}\tilde{\Omega} = (i_{\tilde{v}}d + di_{\tilde{v}})\tilde{w}_{1/2}\, dV^{(5)} \wedge dt = \left(\frac{\partial}{\partial t}\tilde{w}_{1/2} + \frac{\partial}{\partial x_i}(\dot{X}_i \tilde{w}_{1/2}) + \frac{\partial}{\partial p_i}(\tilde{w}_{1/2}\dot{P}_i)\right) dV^{(5)} \wedge dt, \tag{5.24}$$

or employing (5.22) as

$$L_{\tilde{v}}\tilde{\Omega} = -\frac{1}{5!} d\tilde{w}_H^5. \tag{5.25}$$

It is possible to write (5.25) in explicit form

$$\begin{aligned} L_{\tilde{v}}\tilde{\Omega} = -\tfrac{1}{4!2} d\tilde{w}_H^2 \wedge \tilde{w}_H^3 = \; & \frac{1}{2}[\mathcal{E}_i\, d\mathcal{G} \wedge \mathcal{F} \wedge dx_i \wedge dt \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k \\ & -\hat{p}_i\, d\mathcal{G} \wedge \mathcal{F} \wedge \mathcal{F} \wedge dp_i \wedge dt \wedge dp_j \wedge dx_j \\ & -\mathcal{E}_i\, d\mathcal{G} \wedge \mathcal{G} \wedge \mathcal{F} \wedge \mathcal{F} \wedge dx_i \wedge dt]. \end{aligned} \tag{5.26}$$

Matrix valued quantities representing the spin degrees of freedom are introduced in order to describe quantum mechanical particles possessing spin within classical phase space . On the other hand, measure of the related path integral should be a scalar. Thus one needs an appropriate definition of the path integral measure. Basically, there are two choices: To take the trace or the determinant of $\tilde{\Omega}$ over the spin indices. However, due to the fact that $\tilde{w}_{1/2}$ can obviously be written in the form $(I+W)$ in terms of a 2×2 matrix W, its determinant is $\det(I+W) = 1 + \mathrm{Tr}\, W + \frac{1}{2}(\mathrm{Tr}\, W)^2 + \frac{1}{2}\mathrm{Tr}\, W^2$. Thus it will possess a term proportional to $\mathrm{Tr}\, \tilde{w}_{1/2}$. Therefore, one proceeds adopting $\mathrm{Tr}\, \tilde{w}_{1/2}$ as the definition of the related path integral measure. It is possible to show that the Berry curvature given in Appendix (B) is traceless. Using the equality $\mathrm{Tr}\, d\mathcal{G} = d\mathrm{Tr}\, \mathcal{G}$ and (5.26) we get

$$\mathrm{Tr}\,[L_{\tilde{v}}\tilde{\Omega}] = -\frac{1}{4}(d\mathrm{Tr}\,[\mathcal{G} \wedge \mathcal{G}]) \wedge \mathcal{E}_i \mathcal{F} \wedge \mathcal{F} \wedge dx_i \wedge dt.$$

Although it is cumbersome, making use of $\mathcal{G}_{ij}$ presented in Appendix (B) one can show explicitly that

$$-\frac{1}{6}d\mathrm{Tr}\,[\mathcal{G}\wedge\mathcal{G}] = \boldsymbol{\nabla}\cdot\boldsymbol{b}\,dp_1\wedge...dp_5,$$

where $\boldsymbol{b}$ is the 5 dimensional monopole field located at $\boldsymbol{p}=0$:

$$\boldsymbol{b} = \frac{\boldsymbol{p}}{2p^5},\quad \boldsymbol{\nabla}\cdot\boldsymbol{b} = \frac{4\pi^2}{3}\delta^5(p). \tag{5.27}$$

Therefore, one gets

$$\mathrm{Tr}\,[L_{\tilde{v}}\tilde{\Omega}] = -\frac{\pi^2}{2}\delta^5(p)\epsilon^{ijklm}\mathcal{E}_i\mathcal{F}_{jk}\mathcal{F}_{lm}\,dV^{(5)}\wedge dt. \tag{5.28}$$

One should adopt a reduction procedure for the matrix valued velocities in order to obtain the corresponding classical chiral kinetic theory. So that, let $\dot{x}_i, \dot{p}_i$, denote the ordinary velocities and $\sqrt{\mathrm{w}}\equiv\mathrm{Tr}\,[\tilde{w}_{1/2}]$ be the path integral measure. Then, we define $\mathrm{Tr}\,[\dot{X}_i\tilde{w}_{1/2}]\equiv\sqrt{\mathrm{w}}\dot{x}_i$, and $\mathrm{Tr}\,[\tilde{w}_{1/2}\dot{P}_i]\equiv\sqrt{\mathrm{w}}\dot{p}_i$. Therefore, equating (5.24) and (5.28) yields the anomalous Liouville equation in $5+1$ dimensions:

$$\frac{\partial}{\partial t}\sqrt{\mathrm{w}} + \frac{\partial}{\partial x_i}(\sqrt{\mathrm{w}}\dot{x}_i) + \frac{\partial}{\partial p_i}(\sqrt{\mathrm{w}}\dot{p}_i) = -\frac{\pi^2}{2}\delta^5(p)\epsilon^{ijklm}\mathcal{E}_i\mathcal{F}_{jk}\mathcal{F}_{lm}. \tag{5.29}$$

Next step is to link the anomalous expression (5.29) to the non-conservation of particle number. In order to achieve this, introduce the probability distribution $f(x,p,t)$, which satisfies the collisionless Boltzmann equation,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\dot{x}_i + \frac{\partial f}{\partial p_i}\dot{p}_i = 0. \tag{5.30}$$

Once the phase-space probability density is defined as $\rho(x,p,t) = \sqrt{\mathrm{w}}f$, (5.29) and (5.30) lead to the non-conservation of the probability:

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho\dot{x}_i)}{\partial x_i} + \frac{\partial(\rho\dot{p}_i)}{\partial p_i} = -\frac{\pi^2}{2}f\delta^5(p)\epsilon^{ijklm}\mathcal{E}_i\mathcal{F}_{jk}\mathcal{F}_{lm}. \tag{5.31}$$

One introduces the chiral particle number density $n(x,t) = \int\frac{d^5p}{(2\pi)^5}\rho$ and the chiral particle current density $j_i(x,t) = \int\frac{d^5p}{(2\pi)^5}\rho\dot{x}_i$. By integrating (5.31) over the momentum p_i and setting $\int\frac{d^5p}{(2\pi)^5}\frac{\partial(\rho\dot{p}_i)}{\partial p_i} = 0$, as it was discussed in Section 5.3, she/he finds

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot\vec{j} = -\frac{1}{(4\pi)^3}f(x,p=0,t)\epsilon^{ijklm}\mathcal{E}_i\mathcal{F}_{jk}\mathcal{F}_{lm}.$$

Therefore, the chiral current is not conserved.

5.4.3 Chiral magnetic effect in $5+1$ dimensions

To establish the CME in $5+1$ dimensions it is needed to solve the equation of motion (5.21) for $\dot{X}_i\tilde{w}_{1/2}$. One will use method which is presented in Section 5.3. It is demonstrated that solutions of the equations of motion for $\dot{\boldsymbol{X}}$ and $\dot{\boldsymbol{P}}$ can be obtained by inspecting the Lie derivative of the volume form associated with $\tilde{v}$. It is possible to expose $\tilde{\Omega}$ explicitly in order to express $L_{\tilde{v}}\tilde{\Omega}$ in the desired form as

$$\begin{aligned}
\tilde{w}_H^5 &= dp_i \wedge dx_i \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k \wedge dp_l \wedge dx_l \wedge dp_m \wedge dx_m \\
&- 20\mathcal{F}\wedge\mathcal{G}\wedge dp_i \wedge dx_i \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k + 30\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{G}\wedge\mathcal{G}\wedge dp_i \wedge dx_i \\
&- 5\hat{p}_i dp_i \wedge dt \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k \wedge dp_l \wedge dx_l \wedge dp_m \wedge dx_m \\
&- 20\mathcal{E}_i\mathcal{G}\wedge dx_i \wedge dt \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k \wedge dp_l \wedge dx_l \\
&+ 60\hat{p}_i\mathcal{G}\wedge\mathcal{F}\wedge dp_i \wedge dt \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k \\
&+ 60\mathcal{E}_i\mathcal{G}\wedge\mathcal{G}\wedge\mathcal{F}\wedge dx_i \wedge dt \wedge dp_j \wedge dx_j - 30\hat{p}_i\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{G}\wedge\mathcal{G}\wedge dp_i \wedge dt \\
&+ 5\mathcal{E}_i dx_i \wedge dt \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k \wedge dp_l \wedge dx_l \wedge dp_m \wedge dx_m \\
&- 20\hat{p}_i\mathcal{F}\wedge dp_i \wedge dt \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k \wedge dp_l \wedge dx_l \\
&- 60\mathcal{E}_i\mathcal{F}\wedge\mathcal{G}\wedge dx_i \wedge dt \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k \\
&+ 60\hat{p}_i\mathcal{G}\wedge\mathcal{F}\wedge\mathcal{F}\wedge dp_i \wedge dt \wedge dp_j \wedge dx_j + 30\mathcal{E}_i\mathcal{G}\wedge\mathcal{G}\wedge\mathcal{F}\wedge\mathcal{F}\wedge dx_i \wedge dt.
\end{aligned}$$

The Liouville equation (5.25) can now be written in the form

$$\begin{aligned}
L_{\tilde{v}}\tilde{\Omega} = -\frac{1}{5!}d\tilde{w}_H^5 \;&=\; \Bigg[\frac{\partial}{\partial t}\Big(1+\frac{1}{2}\mathcal{F}_{ij}\mathcal{G}_{ij}+\frac{1}{8}\mathcal{F}_{ij}\mathcal{F}_{kl}(\mathcal{G}_{ij}\mathcal{G}_{kl}+\mathcal{G}_{ik}\mathcal{G}_{lj}+\mathcal{G}_{lj}\mathcal{G}_{ik})\Big) \\
&+\frac{\partial}{\partial x_r}\Big(\hat{p}_r+\mathcal{E}_i\mathcal{G}_{ri}+\frac{1}{2}\mathcal{F}_{ij}(\hat{p}_r\mathcal{G}_{ij}+2\hat{p}_j\mathcal{G}_{ri}) \\
&+\frac{1}{4}\mathcal{E}_i\mathcal{F}_{jk}(\mathcal{G}_{ri}\mathcal{G}_{jk}+\mathcal{G}_{jk}\mathcal{G}_{ri}+2\mathcal{G}_{rj}\mathcal{G}_{ki}+2\mathcal{G}_{ki}\mathcal{G}_{rj}) \\
&+\frac{1}{64}\mathcal{F}_{ij}\mathcal{F}_{kl}\mathcal{G}_{mn}\mathcal{G}_{st}\hat{p}_p\epsilon_{rijkl}\epsilon_{mnstp}\Big) \\
&+\frac{\partial}{\partial p_r}\Big(\mathcal{E}_r+\hat{p}_i\mathcal{F}_{ri}+\frac{1}{2}\mathcal{G}_{ij}(\mathcal{E}_r\mathcal{F}_{ij}+2\mathcal{E}_j\mathcal{F}_{ri}) \\
&+\frac{1}{2}\hat{p}_i\mathcal{G}_{jk}(\mathcal{F}_{ri}\mathcal{F}_{jk}+2\mathcal{F}_{rj}\mathcal{F}_{ki}) \\
&+\frac{1}{64}\mathcal{F}_{mn}\mathcal{F}_{st}\mathcal{E}_p\mathcal{G}_{ij}\mathcal{G}_{kl}\epsilon_{rijkl}\epsilon_{mnstp}\Big)\Bigg]dV\wedge dt.
\end{aligned} \tag{5.32}$$

Comparison of (5.32) with (5.24) permits to read directly:

$$
\begin{aligned}
\tilde{w}_{1/2} &= I + \frac{1}{2}\mathcal{F}_{ij}\mathcal{G}_{ij} + \frac{1}{8}\mathcal{F}_{ij}\mathcal{F}_{kl}(\mathcal{G}_{ij}\mathcal{G}_{kl} + \mathcal{G}_{ik}\mathcal{G}_{lj} + \mathcal{G}_{lj}\mathcal{G}_{ik}) \\
\dot{X}_r\tilde{w}_{1/2} &= \hat{p}_r + \mathcal{E}_i\mathcal{G}_{ri} + \frac{1}{2}\mathcal{F}_{ij}(\hat{p}_r\mathcal{G}_{ij} + 2\hat{p}_j\mathcal{G}_{ri}) + \\
&\quad \frac{1}{4}\mathcal{E}_i\mathcal{F}_{jk}(\mathcal{G}_{ri}\mathcal{G}_{jk} + \mathcal{G}_{jk}\mathcal{G}_{ri} + 2\mathcal{G}_{rj}\mathcal{G}_{ki} + 2\mathcal{G}_{ki}\mathcal{G}_{rj}) \\
&\quad + \frac{1}{64}\mathcal{F}_{ij}\mathcal{F}_{kl}\mathcal{G}_{mn}\mathcal{G}_{st}\hat{p}_p\epsilon_{rijkl}\epsilon_{mnstp} \qquad \textbf{(5.33)} \\
\tilde{w}_{1/2}\dot{P}_r &= \mathcal{E}_r + \hat{p}_i\mathcal{F}_{ri} + \frac{1}{2}\mathcal{G}_{ij}(\mathcal{E}_r\mathcal{F}_{ij} + 2\mathcal{E}_j\mathcal{F}_{ri}) + \frac{1}{2}\hat{p}_i\mathcal{G}_{jk}(\mathcal{F}_{ri}\mathcal{F}_{jk} + 2\mathcal{F}_{rj}\mathcal{F}_{ki}) \\
&\quad + \frac{1}{64}\mathcal{F}_{mn}\mathcal{F}_{st}\mathcal{E}_p\mathcal{G}_{ij}\mathcal{G}_{kl}\epsilon_{rijkl}\epsilon_{mnstp}).
\end{aligned}
$$

One will build the the classical current using a phase space probability function $f(x,p,t)$ and then take the trace over the spin indices of the solution (5.33) as

$$
\begin{aligned}
j_r &= \int \frac{d^5p}{(2\pi)^5}\mathrm{Tr}\,[\dot{X}_r\tilde{w}_{1/2}]f \\
&= 2\int \frac{d^5p}{(2\pi)^5}\hat{p}_r f + \frac{1}{2}\int \frac{d^5p}{(2\pi)^5}\mathcal{E}_i\mathcal{F}_{jk}\mathrm{Tr}\,[\mathcal{G}_{ri}\mathcal{G}_{jk} + 2\mathcal{G}_{rj}\mathcal{G}_{ki}]f \qquad \textbf{(5.34)} \\
&\quad + \frac{1}{64}\int \frac{d^5p}{(2\pi)^5}\mathcal{F}_{ij}\mathcal{F}_{kl}\epsilon_{rijkl}\mathrm{Tr}\,[\mathcal{G}_{mn}\mathcal{G}_{st}\hat{p}_p\epsilon_{mnstp}]f.
\end{aligned}
$$

The last term in (5.34) gives the $5+1$ dimensional chiral magnetic effect

$$
j_r^{CME} = -\frac{3}{8}\int \frac{d^5p}{(2\pi)^5}\mathcal{F}_{ij}\mathcal{F}_{kl}\epsilon_{rijkl}(\hat{\boldsymbol{p}}\cdot\boldsymbol{b})f,
$$

where $\boldsymbol{b}$ is the monopole field (5.27). It is possible to perform the angular part of the momentum integral for an isotropic momentum distribution $f(E)$ and obtain

$$
j_r^{CME} = -\frac{1}{8(2\pi)^3}\epsilon_{rijkl}\mathcal{F}_{ij}\mathcal{F}_{kl}\int dE\, f(E).
$$

This is in accord with the chiral magnetic effect proposed in [35]. When the Fermi-Dirac distribution is considered, it yields $j_r^{CME} = (-\mu/8(2\pi)^3)\epsilon_{rijkl}\mathcal{F}_{ij}\mathcal{F}_{kl}$, at finite chemical potential μ.

5.5 Chiral Anomaly and Chiral Magnetic Effect in $d+1$ Dimensions

In this part, it is aimed to generalize the formalism to $2n+2,\ n=1,2...$ dimensions where $n=\frac{d-1}{2}$. In order to achieve this one begins with the definition of the matrix valued symplectic two-form $\tilde{W}_H$ as

$$
\tilde{W}_H \equiv dp_{\mathrm{A}}\wedge dx_{\mathrm{A}} + \mathcal{F} - \mathcal{G} - \hat{p}_{\mathrm{A}}dp_{\mathrm{A}}\wedge dt + \mathcal{E}_{\mathrm{A}}dx_{\mathrm{A}}\wedge dt,
$$

where the capital letter $\mathrm{A} = 1, .., 2n+1$. $\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mathrm{AB}} dx_{\mathrm{A}} \wedge dx_{\mathrm{B}}$ is the Abelian electromagnetic field tensor and $\mathcal{G} = \frac{1}{2}\mathcal{G}_{\mathrm{AB}} dp_{\mathrm{A}} \wedge dp_{\mathrm{B}}$ is the Berry curvature. Electric field pointing towards the $\hat{x}_{\mathrm{A}}$ direction is denoted as $\mathcal{E}_{\mathrm{A}}$. The symplectic two-form $\tilde{W}_H$ is a $2^{n-1} \times 2^{n-1}$ matrix in spin indices which are suppressed for brevity. The time evolution of the $4n+2$ dimensional phase space $(x_{\mathrm{A}}, p_{\mathrm{A}})$ is acquired through the interior product of the matrix valued vector field

$$\tilde{V} = \frac{\partial}{\partial t} + \dot{X}_{\mathrm{A}} \frac{\partial}{\partial x_{\mathrm{A}}} + \dot{P}_{\mathrm{A}} \frac{\partial}{\partial p_{\mathrm{A}}},$$

with the symplectic two-form $\tilde{W}_H$ as

$$i_{\tilde{V}} \tilde{W}_H = 0.$$

Time evolution will yield the following equations of motion:

$$\dot{P}_{\mathrm{A}} = \dot{X}_{\mathrm{B}} \mathcal{F}_{\mathrm{AB}} + \mathcal{E}_{\mathrm{A}}, \tag{5.35a}$$

$$\dot{X}_{\mathrm{A}} = \mathcal{G}_{\mathrm{AB}} \dot{P}_{\mathrm{B}} + \hat{p}_{\mathrm{A}}, \tag{5.35b}$$

Here, $(\dot{X}_{\mathrm{A}}, \dot{P}_{\mathrm{A}})$ are matrices.

5.5.1 Chiral anomaly in $d+1$ dimensions

The phase space volume form is defined as $\tilde{\Omega}$ by

$$\tilde{\Omega} \equiv \frac{(-1)^{n+1}}{(2n+1)!} \tilde{W}_H^{2n+1} \wedge dt. \tag{5.36}$$

and expressed in terms of the $2d$ dimensional Liouville measure dV as

$$\tilde{\Omega} = \tilde{W}_{1/2} dV \wedge dt, \tag{5.37}$$

where $\tilde{W}_{1/2}$ is the Pfaffian of the $(4n+2) \times (4n+2)$ matrix

$$\begin{pmatrix} \mathcal{F}_{\mathrm{AB}} & -\delta_{\mathrm{AB}} \\ \delta_{\mathrm{AB}} & -\mathcal{G}_{\mathrm{AB}} \end{pmatrix}.$$

One needs the Lie derivative of the volume form $\tilde{\Omega}$ in order to derive the Liouville equation. By making use of (5.37) it can be written formally as

$$L_{\tilde{V}} \tilde{\Omega} = (i_{\tilde{V}} d + d i_{\tilde{V}}) \tilde{W}_{1/2}\, dV \wedge dt = \left(\frac{\partial}{\partial t} \tilde{W}_{1/2} + \frac{\partial}{\partial x_{\mathrm{A}}} (\dot{X}_{\mathrm{A}} \tilde{W}_{1/2}) + \frac{\partial}{\partial p_{\mathrm{A}}} (\tilde{W}_{1/2} \dot{P}_{\mathrm{A}}) \right) dV \wedge dt. \tag{5.38}$$

On the other hand, in order to acquire Liouville equation explicitly the definition (5.36) should be employed,

$$L_{\tilde{V}}\tilde{\Omega} = \frac{(-1)^{n+1}}{(2n+1)!} d\tilde{W}_{H}^{2n+1}. \tag{5.39}$$

Among the several terms in $\tilde{W}_{H}^{2n+1}$, the chiral anomaly stems from the one which includes the singularity in momentum space:

$$\frac{(-1)^{n}(2n+1)!}{(n!)^2} \overbrace{\mathcal{G}...\mathcal{G}}^{n\ times}\, \mathcal{E}_{\mathrm{A}} \overbrace{\mathcal{F}...\mathcal{F}}^{n\ times} dx_{\mathrm{A}} \wedge dt. \tag{5.40}$$

In Appendix (B) it is also briefly reported, following [65], how the singularity is calculated to be

$$\begin{aligned} \mathrm{Tr}\,[d(\overbrace{\mathcal{G}...\mathcal{G}}^{n\ times})] &= \frac{1}{2^n}\frac{\partial}{\partial p_{\mathrm{A}}} \mathrm{Tr}\,[\epsilon_{\mathrm{ABC...DE}} \overbrace{\mathcal{G}_{\mathrm{BC}}...\mathcal{G}_{\mathrm{DE}}}^{n\ times}] d^{2n+1}p \\ &= \frac{(-1)^{n+1}(2n)!}{2^n} (\boldsymbol{\nabla}\cdot\boldsymbol{b}) d^{2n+1}p, \end{aligned}$$

where $\boldsymbol{b}$ is the $2n+1$ dimensional Dirac monopole field

$$\boldsymbol{b} = \frac{\boldsymbol{p}}{2p^{2n+1}}. \tag{5.41}$$

Following [65] one calculates the singularity to be

$$\begin{aligned} \mathrm{Tr}\,[d(\overbrace{\mathcal{G}...\mathcal{G}}^{n\ times})] &= \frac{1}{2^n}\frac{\partial}{\partial p_{\mathrm{A}}} \mathrm{Tr}\,[\epsilon_{\mathrm{ABC...DE}} \overbrace{\mathcal{G}_{\mathrm{BC}}...\mathcal{G}_{\mathrm{DE}}}^{n\ times}] d^{2n+1}p \\ &= \frac{(-1)^{n+1}(2n)!}{2^n} (\boldsymbol{\nabla}\cdot\boldsymbol{b}) d^{2n+1}p, \end{aligned}$$

where $\boldsymbol{b}$ is the $2n+1$ dimensional Dirac monopole field

$$\boldsymbol{b} = \frac{\boldsymbol{p}}{2p^{2n+1}}. \tag{5.42}$$

As being a monopole field (5.42) satisfies

$$\boldsymbol{\nabla}\cdot\boldsymbol{b} = \frac{\mathrm{Vol}(S^{2n})}{2}\delta^{2n+1}(p),$$

where $\mathrm{Vol}(S^{2n})$ denotes volume of the $2n$-sphere. The details of the calculation is briefly saved for Appendix (B). Hence, one attains

$$\mathrm{Tr}\,[L_{\tilde{V}}\tilde{\Omega}] = \frac{(-1)^{n+1}(2n)!}{(n!)^2\, 2^{2n+1}} \mathrm{Vol}(S^{2n})\delta^{2n+1}(p)\epsilon_{\mathrm{ABC...DE}}\mathcal{E}_{\mathrm{A}} \overbrace{\mathcal{F}_{\mathrm{BC}}...\mathcal{F}_{\mathrm{DE}}}^{n\ times} dV \wedge dt. \tag{5.43}$$

However, within the semiclassical approximation one defines

$$\sqrt{W} \equiv \mathrm{Tr}\,[\tilde{W}_{1/2}],\ \mathrm{Tr}\,[(\dot{X}_{\mathrm{A}}\tilde{W}_{1/2}] \equiv \sqrt{W}\dot{x}_{\mathrm{A}},\ \mathrm{Tr}\,[\tilde{W}_{1/2}\dot{P}_{\mathrm{A}})] \equiv \sqrt{W}\dot{p}_{\mathrm{A}},$$

where $(\dot{x}_{\mathrm{A}}, \dot{p}_{\mathrm{A}})$ denote the classical velocities. Equating the trace of (5.38) with (5.43) the semiclassical anomalous Liouville equation is obtained as

$$\left(\frac{\partial}{\partial t}\sqrt{W} + \frac{\partial}{\partial x_{\mathrm{A}}}(\sqrt{W}\dot{x}_{\mathrm{A}}) + \frac{\partial}{\partial p_{\mathrm{A}}}(\sqrt{W}\dot{p}_{\mathrm{A}})\right) =$$
$$\frac{(-1)^{n+1}(2n)!}{(n!)^2\, 2^{2n+1}}\mathrm{Vol}(S^{2n})\delta^{2n+1}(p)\epsilon_{\mathrm{ABC\ldots DE}}\mathcal{E}_{\mathrm{A}}\overbrace{\mathcal{F}_{\mathrm{BC}}\ldots\mathcal{F}_{\mathrm{DE}}}^{n\ times}.$$

It is possible to link it to the non-conservation of the chiral particle number by employing the phase space distribution $f(x, p, t)$ which satisfies the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_{\mathrm{A}}}\dot{x}_{\mathrm{A}} + \frac{\partial f}{\partial p_{\mathrm{A}}}\dot{p}_{\mathrm{A}} = 0,$$

and define the probability density by $\rho(x, p, t) = \sqrt{W}f$. Thus one observes

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\dot{x}_{\mathrm{A}})}{\partial x_{\mathrm{A}}} + \frac{\partial(\rho\dot{p}_{\mathrm{A}})}{\partial p_{\mathrm{A}}} = \frac{(-1)^{n+1}(2n)!}{(n)!^2\, 2^{2n+1}}\mathrm{Vol}(S^{2n})f\delta^{2n+1}(p)\epsilon_{\mathrm{ABC\ldots DE}}\mathcal{E}_{\mathrm{A}}\overbrace{\mathcal{F}_{\mathrm{BC}}\ldots\mathcal{F}_{\mathrm{DE}}}^{n\ times}, \tag{5.44}$$

which is the non-conservation of the phase space probability. Then, introducing the chiral particle density $n(x, t) = \int \frac{d^{2n+1}}{(2\pi)^{2n+1}}\rho$ and the chiral current density $j_{\mathrm{A}} = \int \frac{d^{2n+1}}{(2\pi)^{2n+1}}\rho\dot{x}_{\mathrm{A}}$, it is possible to establish the non-conservation of the chiral current as

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot\vec{j} = \frac{(-1)^{n+1}}{n!\, 2^n\, (2\pi)^{n+1}}f(x, p = 0, t)\mathcal{E}_{\mathrm{A}}\overbrace{\mathcal{F}_{\mathrm{BC}}\ldots\mathcal{F}_{\mathrm{DE}}}^{n\ times}. \tag{5.45}$$

It is derived by integrating (5.44) over the momentum space and setting $\mathrm{Vol}(S^{2n}) = \frac{2^{2n+1}\pi^n n!}{(2n)!}$. (5.45) is the semiclassical manifestation of the chiral anomaly in any even dimensions.

5.5.2 Chiral magnetic effect in $d + 1$ dimensions

The solutions of the equations of motion (5.35) for $(\dot{X}_{\mathrm{A}}\tilde{W}_{1/2}, \tilde{W}_{1/2}\dot{P}_{\mathrm{A}})$ can be directly provided in terms of the phase space variables $(x_{\mathrm{A}}, p_{\mathrm{A}})$, by equating the right hand sides of (5.38) and (5.39):

$$\frac{(-1)^{n+1}}{(2n+1)!}d\tilde{W}_H^{2n+1} = \frac{\partial}{\partial t}\tilde{W}_{1/2} + \frac{\partial}{\partial x_{\mathrm{A}}}(\dot{X}_{\mathrm{A}}\tilde{W}_{1/2}) + \frac{\partial}{\partial p_{\mathrm{A}}}(\tilde{W}_{1/2}\dot{P}_{\mathrm{A}}).$$

Dealing with $4n+2$ dimensional phase space, $\dot{X}_{\mathrm{A}}\tilde{W}_{1/2}$ has many terms including

$$\begin{aligned}
\dot{X}_{\mathrm{A}}\tilde{W}_{1/2} &= \frac{(-1)^n}{(2n)!}dx_{\mathrm{A}}\wedge\hat{p}_{\mathrm{B}}dp_{\mathrm{B}}\wedge dt\wedge\overbrace{dp_{\mathrm{C}}\wedge dx_{\mathrm{C}}...dp_{\mathrm{D}}\wedge dx_{\mathrm{D}}}^{2n\ times} \\
&+\frac{(-1)^n}{(2n-1)!}dx_{\mathrm{A}}\wedge\mathcal{E}_{\mathrm{B}}dx_{\mathrm{B}}\wedge dt\wedge\mathcal{G}\overbrace{dp_{\mathrm{C}}\wedge dx_{\mathrm{C}}...dp_{\mathrm{D}}\wedge dx_{\mathrm{D}}}^{2n-1\ times} \\
&+\frac{(-1)^{n+1}}{(2n-2)!}dx_{\mathrm{A}}\wedge\hat{p}_{\mathrm{B}}dp_{\mathrm{B}}\wedge dt\wedge\mathcal{G}\mathcal{F}\overbrace{dp_{\mathrm{C}}\wedge dx_{\mathrm{C}}...dp_{\mathrm{D}}\wedge dx_{\mathrm{D}}}^{2n-2\ times} \\
&+...+\frac{1}{(n!)^2\,2^{2n}}\epsilon_{\mathrm{ABC...DE}}\overbrace{\mathcal{F}_{\mathrm{BC}}...\mathcal{F}_{\mathrm{DE}}}^{n\ times}\epsilon_{\mathrm{IJK...LM}}\hat{p}_{\mathrm{I}}\overbrace{\mathcal{G}_{\mathrm{JK}}...\mathcal{F}_{\mathrm{LM}}}^{n\ times}. \qquad \textbf{(5.46)}
\end{aligned}$$

One defines the current within the semiclassical approximation as

$$j_{\mathrm{A}} = \int\frac{d^{2n+1}p}{(2\pi)^{2n+1}}\mathrm{Tr}\,[\dot{X}_{\mathrm{A}}\tilde{W}^{\mathrm{CME}}_{1/2}]f(x,p,t).$$

One deals with the CME which is generated by the terms depending on the external magnetic field $\mathcal{F}_{\mathrm{AB}}$. As it is denoted in Appendix (B), once the trace is taken over the spin indices depending on the external magnetic field $\mathcal{F}_{\mathrm{AB}}$, only one term survives which is generated by the last term given in (5.46). Therefore the chiral magnetic current is exactly found to be

$$\begin{aligned}
j^{\mathrm{CME}}_{\mathrm{A}} &= \frac{1}{2^{2n}(n!)^2}\int\frac{d^{2n+1}p}{(2\pi)^{2n+1}}\epsilon_{\mathrm{ABC...DE}}\overbrace{\mathcal{F}_{\mathrm{BC}}...\mathcal{F}_{\mathrm{DE}}}^{n\ times}\mathrm{Tr}\,[\epsilon_{\mathrm{IJK...LM}}\hat{p}_{\mathrm{I}}\overbrace{\mathcal{G}_{\mathrm{JK}}...\mathcal{G}_{\mathrm{LM}}}^{n\ times}]f(x,p,t) \\
&= \frac{(-1)^{n+1}(2n)!}{2^{2n}(n!)^2}\int\frac{d^{2n+1}p}{(2\pi)^{2n+1}}\epsilon_{\mathrm{ABC...DE}}\overbrace{\mathcal{F}_{\mathrm{BC}}...\mathcal{F}_{\mathrm{DE}}}^{n\ times}(\hat{\boldsymbol{p}}\cdot\boldsymbol{b})f(x,p,t), \qquad \textbf{(5.47)}
\end{aligned}$$

where $\boldsymbol{b}$ is the Dirac monopole field given in (5.42). When an isotropic momentum distribution $f=f(E)$, is considered the angular part of (5.47) can be calculated, so that one establishes the chiral magnetic current as

$$\begin{aligned}
j^{\mathrm{CME}}_{\mathrm{A}} &= \frac{(-1)^{n+1}(2n)!}{2^{2n}(n!)^2}\frac{\mathrm{Vol}(S^{2n})}{2(2\pi)^{2n+1}}\epsilon_{\mathrm{ABC...DE}}\overbrace{\mathcal{F}_{\mathrm{BC}}...\mathcal{F}_{\mathrm{DE}}}^{n\ times}\int dEf(E), \\
&= \frac{(-1)^{n+1}}{2^n(2\pi)^{n+1}n!}\epsilon_{\mathrm{ABC...DE}}\overbrace{\mathcal{F}_{\mathrm{BC}}...\mathcal{F}_{\mathrm{DE}}}^{n\ times}\int dEf(E).
\end{aligned}$$

This is the chiral magnetic current conjectured in [35].

5.6 Discussions

The non-Abelian Berry gauge field arising from the $5 + 1$ dimensional Weyl Hamiltonian is calculated explicitly and it is demonstrated that the Berry field strength yields a monopole situated at the origin of phase space. In fact in any even $d + 1$ dimensions the related Berry field strength engenders a Dirac monopole field. This monopole field is responsible for the chiral anomaly manifested itself in the kinetic theory of electrons as non-conservation of particle current. It is shown that this monopole is also the source of the CME. An efficient method of finding the path integral measure is presented and solutions of the equations of motion for the first derivatives of the phase space variables weighted by this measure. This furnished the possibility of obtaining the chiral current directly inspecting Liouville equation. Hence the CME and chiral anomaly is calculated in any even dimensions within the same formulation. The accomplished CME is in accord with the one conjectured in [35]. The main novelty is to keep the spin dependence explicit without attributing them some dynamical variables.

A semiclassical study of the massive Dirac particle engaging the non-Abelian Berry gauge fields was presented in [66]. There some dynamical variables have been associated with the spin degrees of freedom. In principle the approach of dealing with the non-Abelian Berry gauge fields presented here can be adopted to perform a similar study in the ordinary classical phase space without enlarging it with some new dynamical variables.

The formalism based on the matrix valued symplectic form is not restricted to even dimensions. It can be employed to establish solution of the equations of motion in terms of phase space variables in any dimensions. These solutions which exhibit the spin dependence explicitly can be useful to formulate some interesting physical phenomena like the spin Hall effect.

6. TOPOLOGICAL CONCEPTS RELATED TO THE WEYL HAMILTONIANS AND THE BERRY GAUGE FIELD

6.1 Purpose

Physical systems may have nontrivial topological and geometrical properties. Besides their interesting mathematical structures, these properties yield important physical consequences. Magnetic monopole in 3 dimensional Euclidean space $\boldsymbol{R}^3$ is the simplest example where the existence of the monopole makes the space homotopic to the 2-sphere S^2. The gauge transformations on S^2 are classified by the first homotopy group and this group is found to be isomorphic to the set of integers, $\Pi_1(S^1) \simeq \boldsymbol{Z}$. This integer is known as the winding number of the gauge transformations as it counts the number of times the gauge transformation wind around the gauge group $U(1)$. The winding number can be expressed in terms of the topological first Chern number. Another example that has numerous applications in quantum mechanics and condensed matter theory is the Berry phase which is the nontrivial phase factor due to the adiabatic evolution of the eigenstates in the parameter space of the Hamiltonian. The Berry phase is the holonomy associated to the $U(1)$ bundle. Topological and geometrical concepts also play an important role in quantum field theories. For example, the axial or chiral anomaly is the violation of the classically conserved chiral current at the quantum level. The expectation value of the divergence of the current results in the difference between the zero modes of opposite chirality which is the analytic index of the Weyl operator. Because of the Atiyah-Singer index theorem the analytic index is equal to the topological index which is given in terms of the Chern character [38].

Within the homotopy theory the natural way for an investigation of the physical system is the use of fermionic Green's function $G(w, \boldsymbol{p})$ and its winding number. The winding number is given as an integer valued integral in terms of the Green's function. It reflects some properties of the related Weyl particles [39]. For example, in odd space-time dimensions effective field theory of the time reversal invariant topological insulators is the Chern-Simons action whose coefficient corresponds to the conductivity of the system. This coefficient, hence the conductivity is quantized as it is equal to the winding number of the fermion propagator [15, 52]. In [15, 52] it was also shown that the winding number is equal to the integral of the related Chern character given in terms of the Berry field strength. The same argument is valid in the case of the spin Hall effect [67]. A possible condensed matter realization of Weyl Hamiltonian, namely Weyl semimetal is proposed [40] where the Berry phase reflects the nontrivial topological properties of its band structure.

Chiral anomaly is the nonconservation of the classically conserved chiral current once the theory is quantized. However, in [31, 32] it is shown that a realization of chiral anomaly is possible in the context of semiclassical chiral kinetic theory by introducing the Berry gauge field induced through the diagonalization of the Weyl Hamiltonian. Moreover, a generalization for higher even dimensional spacetimes was achieved in [34]. [31, 32] also acquires the relation between the chiral magnetic effect and chiral anomaly in $3+1$ dimensions. In [68] both the chiral anomaly and the chiral magnetic effect is formulated in terms of differential forms for all even $d+1$ dimensions.

Hence, it is desirable to investigate the topological and geometrical concepts and the relations between them in the context of one particle Weyl systems including Berry gauge fields in any even $d+1$ dimensions. In momentum space, considering the $d+1$ dimensional Weyl Hamiltonian, we calculate the winding number of the fermion propagator which is a topological invariant [39], explore its interrelation with the other topological invariants and search its physical meaning.

6.2 Weyl Hamiltonian, the Berry Gauge Field and the Winding Number

In even $d+1$ dimensional spacetime, the one particle Weyl Hamiltonian

$$\mathcal{H}_{\mathrm{W}} = \boldsymbol{\Sigma} \cdot \boldsymbol{p}, \qquad \textbf{(6.1)}$$

is expressed in terms of the d dimensional momentum vector $\boldsymbol{p}$ and the $2^{\frac{d-1}{2}} \times 2^{\frac{d-1}{2}}$ dimensional $\boldsymbol{\Sigma}$ matrices. $\boldsymbol{\Sigma}$ matrices satisfy the relations $\{\Sigma_{\mathrm{M}}, \Sigma_{\mathrm{N}}\} = 2\delta_{\mathrm{MN}}$ where $M, N = 1, .., d$. The Weyl Hamiltonian (6.1) is derived from the massless Dirac Hamiltonian,

$$\mathcal{H}_D = \boldsymbol{\alpha} \cdot \boldsymbol{p}, \quad \{\alpha_{\mathrm{M}}, \alpha_{\mathrm{N}}\} = 2\delta_{\mathrm{MN}},$$

which is block diagonal in the chiral representation of the $2^{\frac{d+1}{2}} \times 2^{\frac{d+1}{2}}$ dimensional $\boldsymbol{\alpha}$ matrices. One can solve the Weyl equation by the eigenvectors $|\psi_{\pm}^{(\alpha)}(\boldsymbol{p})\rangle$ as,

$$\mathcal{H}_{\mathrm{W}}|\psi_{\pm}^{(\alpha)}(\boldsymbol{p})\rangle = \pm p|\psi_{\pm}^{(\alpha)}(\boldsymbol{p})\rangle,$$

where $\alpha, \beta ... = 1, .., \frac{d-1}{2}$ indicating the $\frac{d-1}{2}$ fold degeneracy of the each eigenvalue $\pm p = \pm|\boldsymbol{p}|$. One can find a unitary matrix U which diagonalizes (6.1) as

$$U\mathcal{H}_{\mathrm{W}}U^{\dagger} = \mathrm{diag}(p, -p) = p(\mathcal{I}^{+} - \mathcal{I}^{-}). \tag{6.2}$$

$\mathcal{I}^{+}$ and $\mathcal{I}^{-}$ are $2^{\frac{d-1}{2}} \times 2^{\frac{d-1}{2}}$ dimensional matrices projecting onto the positive and negative energy subspaces respectively:

$$\mathcal{I}^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{I}^{-} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.3}$$

In terms of U the Berry gauge field is expressed as

$$\mathcal{A} = i\mathcal{I}^{+}U\partial_{\boldsymbol{p}}U^{\dagger}\mathcal{I}^{+}. \tag{6.4}$$

(6.4) is Abelian for $3+1$ dimensional spacetime however as a result of the $\frac{d-1}{2}$ fold degeneracy, it will be non-Abelian in higher dimensions. Equivalently, we can write the Berry gauge field also in terms of the positive energy solutions:

$$\mathcal{A}_{\mathrm{M}}^{\alpha\beta} = i\langle\psi_{+}^{\alpha}|\partial_{p_{\mathrm{M}}}|\psi_{+}^{\beta}\rangle.$$

The related field strength is defined as

$$\mathcal{G}_{\mathrm{MN}}^{\alpha\beta} = \partial_{\mathrm{M}}\mathcal{A}_{\mathrm{N}}^{\alpha\beta} - \partial_{\mathrm{N}}\mathcal{A}_{\mathrm{M}}^{\alpha\beta} - i[\mathcal{A}_{\mathrm{M}}, \mathcal{A}_{\mathrm{N}}]^{\alpha\beta}, \tag{6.5}$$

where we used the shorthand notation $\partial_{\mathrm{M}} \equiv \partial_{p_{\mathrm{M}}}$.

On the other hand, the $d+1$ dimensional winding number C_d is an integer valued topological invariant which is defined in the momentum space of the system as an integral in terms of the Green's function $G(w, \boldsymbol{p})$ and its inverse $G^{-1}(w, \boldsymbol{p})$ [39]:

$$C_d = N_d \int d^d p \, dw \, \epsilon^{\mu\nu..\rho}\mathrm{Tr}\,[(G\partial_{p_\mu}G^{-1})(G\partial_{p_\nu}G^{-1})...(G\partial_{p_\rho}G^{-1})]. \tag{6.6}$$

Here, $\mu, \nu = 0, ..., d$ and $\epsilon^{\mu\nu..\rho}$ is the $d+1$ dimensional totally antisymmetric Levi-Civita tensor. N_d is the normalization constant which depends on the dimension d. (6.6) is not effected under the infinitesimal change $G \to G + \delta G$ which reflects its topological invariance. This property becomes invaluable if there exists a physical quantity corresponding to (6.6).

In order to write $G(w,\boldsymbol{p})$ we invert the relation (6.2)

$$\mathcal{H}_{\mathrm{W}} = p(P^+ - P^-), \quad P^\pm = U^\dagger \mathcal{I}^\pm U, \tag{6.7}$$

where $P^\pm$ are the projection operators:

$$P^+ + P^- = 1, \quad P^\pm P^\mp = 0, \quad P^\pm P^\pm = P^\pm. \tag{6.8}$$

Now, it is possible to express $G(w,\boldsymbol{p})$ and its inverse $G^{-1}(w,\boldsymbol{p})$ in terms of $P^\pm$ as

$$G(w,\boldsymbol{p}) = \frac{P^+}{w-p} + \frac{P^-}{w+p}, \quad G^{-1}(w,\boldsymbol{p}) = w - p(P^+ - P^-). \tag{6.9}$$

The derivatives of $G^{-1}(w,\boldsymbol{p})$ with respect to $(w,\boldsymbol{p})$ are computed as

$$\frac{\partial G^{-1}}{\partial w} = 1, \quad \frac{\partial G^{-1}}{\partial p_{\mathrm{M}}} = -\left(\frac{p^{\mathrm{M}}}{p}(P^+ - P^-) + p\partial_{\mathrm{M}}(P^+ - P^-)\right). \tag{6.10}$$

6.3 $3+1$ Dimensional Weyl Hamiltonian and the Winding Number

In $3+1$ dimensional spacetime the Weyl Hamiltonian $\mathcal{H}^3_{\mathrm{W}}$ is given in terms of the Pauli spin matrices σ_i:

$$\mathcal{H}^3_{\mathrm{W}} = \boldsymbol{\sigma}\cdot\boldsymbol{p} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}. \tag{6.11}$$

By means of the 2×2 unitary matrix

$$U = \begin{pmatrix} N_+ & \frac{N_+(p-p_3)}{p_1+ip_2} \\ N_- & \frac{-N_-(p+p_3)}{p_1+ip_2} \end{pmatrix}, \quad N_+ = \sqrt{\left(\tfrac{p+p_3}{2p}\right)}, N_- = \sqrt{\left(\tfrac{p-p_3}{2p}\right)}, \tag{6.12}$$

one can diagonalize the $3+1$ dimensional Weyl Hamiltonian (6.11) as

$$U\mathcal{H}^3_{\mathrm{W}}U^\dagger = \mathrm{diag}(p,-p) = p(\mathcal{I}^+ - \mathcal{I}^-),$$

where $\mathcal{I}^+$ and $\mathcal{I}^-$ are defined as in (6.3). It is possible to construct the matrix (6.12) with the solutions $|\psi_\pm(p)\rangle$ of the momentum space eigenvalue equation

$$\mathcal{H}^3_{\mathrm{W}}|\psi_\pm\rangle = \pm p|\psi_\pm\rangle,$$

as

$$U = \big(|\psi_+(p)\rangle\ |\psi_-(p)\rangle\big)^\dagger.$$

The spectral decomposition of (6.11) is given in terms of the projection operators $P^{\pm}$ by inverting the relation (6.2):

$$\mathcal{H}^3_{\rm w} = pU^{\dagger}(\mathcal{I}^+ - \mathcal{I}^-)U = p(P^+ - P^-).$$

One can compute P^+ explicitly by using (6.3) and (6.12) as

$$P^+ = \frac{1}{2p}\begin{pmatrix} p+p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p - p_3 \end{pmatrix}.$$

Utilizing the definitions (6.7) and (6.8) it is possible to express P^+ by means of $\mathcal{H}^3_{\rm w}$:

$$P^+ = \frac{1}{2}(\frac{\mathcal{H}^3_{\rm w}}{p} + 1). \qquad \textbf{(6.13)}$$

Although the explicit forms of the unitary matrix U (6.12) and the projection operator P^+ are presented for completeness, in order to calculate the winding number one only needs the definition of P^+ in terms of the Weyl Hamiltonian as in (6.13). Recalling (6.6), the $3+1$ dimensional winding number $\mathcal{C}_3$ is calculated by using (6.9) and (6.10) as

$$\begin{aligned} \mathcal{C}_3 &= \frac{1}{8\pi^2}\int d^3p\, dw\, \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\,[(G\partial_\mu G^{-1})(G\partial_\nu G^{-1})(G\partial_\rho G^{-1})(G\partial_\sigma G^{-1})] \\ &= \frac{1}{2\pi^2}\int d^3p\, dw\, \epsilon^{abc}\mathrm{Tr}\,[(G^2\partial_a G^{-1})(G\partial_b G^{-1})(G\partial_c G^{-1})], \qquad \textbf{(6.14)} \end{aligned}$$

where $\mu, \nu ... = 0, .., 3$, $a, b, c = 1, 2, 3$ and Tr denotes the trace over the spin indices. It is straightforward to observe that the quadratic and the cubic terms in p_a vanish due to the antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$. A detailed investigation shows that the terms linear in p_a also give a vanishing contribution after the w integration. Hence performing the w integral and using the properties (6.8), the winding number (6.14) is found to be:

$$\mathcal{C}_3 = -\frac{i}{2\pi}\int d^3p\, \epsilon^{abc}\mathrm{Tr}\,[\partial_a P^+ \partial_b P^+ \partial_c P^+]. \qquad \textbf{(6.15)}$$

(6.15) does not vanish under the antisymmetry of the Levi -Civita tensor because P^+ is a 2×2 matrix. However, the integrand is a total derivative,

$$\int d^3p\, \epsilon^{abc}\mathrm{Tr}\,[\partial_a P^+ \partial_b P^+ \partial_c P^+] = \int d^3p\, \boldsymbol{\nabla}\cdot \boldsymbol{K}_3,$$

where K^a_3 is introduced as

$$K^a_3 = \epsilon^{abc}\mathrm{Tr}\,[P^+\partial_b P^+ \partial_c P^+]. \qquad \textbf{(6.16)}$$

It is possible to use the definition (6.13) in (6.16) to express K_3^a in a simple form,

$$\begin{aligned} K_3^a &= \frac{1}{(2p)^3}\epsilon^{abc}\mathrm{Tr}\,[\mathcal{H}_{\mathrm{W}}^3(\partial_b\mathcal{H}_{\mathrm{W}}^3)(\partial_c\mathcal{H}_{\mathrm{W}}^3)] \\ &= \frac{1}{(2p)^3}\epsilon^{abc}\mathrm{Tr}\,[\boldsymbol{\sigma}\cdot\boldsymbol{p}\sigma_b\sigma_c] \\ &= \frac{ip^a}{(2p)^3}2!\mathrm{Tr}\,[1_{2\times 2}], \end{aligned} \quad \textbf{(6.17)}$$

where the $SU(2)$ algebra and trace properties of the Pauli spin matrices are employed. Therefore one obtains K_3^a as

$$K_3^a = \frac{ip^a}{2p^3},$$

which is the field of a monopole with a unit charge located at $p = 0$, that is $\boldsymbol{b}_3 = \frac{\boldsymbol{p}}{2p^3}$. Hence, we conclude that the winding number (6.14) is the divergence of the field $\boldsymbol{b}_3$:

$$\mathcal{C}_3 = \frac{1}{2\pi}\int d^3p\, \boldsymbol{\nabla}\cdot\boldsymbol{b}_3.$$

Using $\boldsymbol{\nabla}\cdot\boldsymbol{b}_3 = 2\pi\delta^3(p)$, we observe that the winding number $\mathcal{C}_3$ is equal to the unit charge of the monopole:

$$\mathcal{C}_3 = \frac{1}{2\pi}\int d^3p\, \boldsymbol{\nabla}\cdot(\frac{\boldsymbol{p}}{2p^3}) = \frac{1}{2\pi}\int d^3p\, \boldsymbol{\nabla}\cdot\boldsymbol{b}_3 = 1. \quad \textbf{(6.18)}$$

(6.14) is a topological invariant, therefore this monopole possesses a topological origin. In [39], it was stated that the winding number (6.14) denotes the chirality of the particle. Thus one concludes that the chirality defines the strength of the monopole. Indeed, if the opposite chirality Hamiltonian $-\boldsymbol{\sigma}\cdot\boldsymbol{p}$ was chosen instead of (6.11), one would have obtained $\mathcal{C}_3 = -1$. On the other hand, the calculation of (6.14) can be performed via the negative energy eigenspace as well. Then, one defines the Berry gauge field in terms of the negative energy solution $|\psi_-(p)\rangle$ and uses the negative eigenspace projector P^-. The value of (6.11) will not change under these changes as it is expected.

Yet (6.17) deserves a closer look. Definition of the projection operators (6.7) enables us to express K_3^a by means of the diagonalization matrix U as

$$K_3^a = \epsilon^{abc}\mathrm{Tr}\,[\mathcal{I}^+(\partial_b U)(\partial_c U^\dagger)\mathcal{I}^+].$$

The Abelian Berry gauge field is computed either in terms of the positive energy solution $|\psi_+\rangle$ or in terms of U (6.12) as

$$\mathcal{A}^a = i\langle\psi^+|\frac{\partial}{\partial p_a}|\psi^+\rangle = i\mathcal{I}^+U\frac{\partial}{\partial p_a}U^\dagger\mathcal{I}^+ = \frac{\epsilon^{ab3}p_b}{2p(p+p_3)}. \quad \textbf{(6.19)}$$

The related Berry field strength is calculated as,

$$\mathcal{G}_{ab} = \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a = i\mathcal{I}^+\big((\partial_a U)(\partial_b U^\dagger) - (\partial_b U)(\partial_a U^\dagger)\big)\mathcal{I}^+.$$

One concludes that

$$K_3^a = \frac{1}{2i}\epsilon^{abc}\mathcal{G}_{bc},$$

which reveals the relation between the winding number (6.14) and the Berry field strength:

$$\mathcal{C}_3 = -\frac{1}{4\pi}\int d^3p\, \epsilon^{abc}\partial_a \mathcal{G}_{bc}.$$

It was known that Berry curvature yields a monopole field $\epsilon^{abc}\mathcal{G}_{bc} = -\frac{p^a}{p^3}$ that is located at $p = 0$ and this field is responsible for the chiral magnetic effect denoted in [31,32,68] and the semiclassical chiral anomaly in 3 + 1 dimensions [31, 32, 34, 68]. Hence, it appears that both phenomena are related to the topological invariant (6.14) which is calculated in the momentum space. Using differential forms, on the 2-sphere S^2 which is the boundary of the 3-ball B^3, this relation turns out to be,

$$\mathcal{C}_3 = -\frac{1}{4\pi}\int_{B^3} d^3p\, \epsilon^{abc}\partial_a \mathcal{G}_{bc} = -\frac{1}{4\pi}\int_{S^2} d^2p\, \epsilon^{\mathrm{bc}}\mathcal{G}_{\mathrm{bc}}, \qquad \textbf{(6.20)}$$

in which b, c represent the polar and the azimuthal angles θ, ϕ respectively and $\mathcal{G}_{\theta\phi} = \frac{\sin\theta}{2}$. (6.20) is the topological first Chern number multiplied with minus one. Over a compact manifold like S^2 the integral of the Berry curvature (Chern character) has to be a quantized number and we calculate it as -1. This equivalence of the Chern number with the winding number was also stated in [15, 52] in the context of massive Dirac Hamiltonian in odd spacetime dimensions.

In the calculation of the first Chern number, one uses the Berry gauge field that is projected onto the positive energy eigenstate (6.19). As it was mentioned that the value of $\mathcal{C}_3$ will not effected whether it is calculated in the negative energy eigenspace. However, calculation of the first Chern number in the negative eigenspace results in $+1$. So that, unlike the winding number $\mathcal{C}_3$, the Chern number is sensitive to the projection.

One would like to emphasize the gauge field structure of the Dirac monopole (6.18), so that she/he defines the 1-form gauge field $\mathcal{B}_3$ as

$$\mathcal{B}_3 = \mathcal{A}_a dp^a. \qquad \textbf{(6.21)}$$

$\mathcal{A}_a$ is the Berry gauge field given in (6.19). $\mathcal{B}_3$ is defined on the upper hemisphere of S^2 as it is singular on the negative z axis. (6.19) is also equal to the gauge field of the $U(1)$ Dirac magnetic monopole in $\boldsymbol{R}^3$ where quantization of the magnetic charge is given in terms of the winding number of the gauge group. However, in this case one is dealing with a monopole of a unit charge emerging from the diagonalization of the Weyl Hamiltonian (6.11). It is found that the winding number of the fermion propagator and the Chern number are equal to this unit charge. The gauge group of $\mathcal{B}_3$ is $U(1)$ and the Chern number (6.20) is known as the winding number of the principal bundle $P(S^2, U(1))$.

6.4 $5+1$ Dimensional Weyl Hamiltonian and the Winding Number

The $5+1$ dimensional Weyl Hamiltonian $\mathcal{H}^5_{\mathrm{W}}$ is a 4×4 matrix expressed in terms of the 5 dimensional momentum vector $\boldsymbol{p}$ and the $3+1$ dimensional Weyl Hamiltonian $\mathcal{H}^3_{\mathrm{W}}$ given in (6.11) as

$$\mathcal{H}^5_{\mathrm{W}} = \boldsymbol{\Sigma}\cdot\boldsymbol{p} = \begin{pmatrix} \mathcal{H}^3_{\mathrm{W}} & i(p_4+ip_5) \\ -i(p_4-ip_5) & -\mathcal{H}^3_{\mathrm{W}} \end{pmatrix}. \tag{6.22}$$

$\boldsymbol{\Sigma}$ matrices are extensions of the Pauli spin matrices to 5 dimensions:

$$\Sigma_a = \begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix};\ {\scriptstyle a=1,2,3,} \quad \Sigma_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \Sigma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

One diagonalizes the $5+1$ dimensional Weyl Hamiltonian (6.22) as

$$U\mathcal{H}^5_{\mathrm{W}}U^\dagger = \mathrm{diag(p,-p)} = \mathrm{p}(\mathcal{I}^+ - \mathcal{I}^-),$$

where $\mathcal{I}^\pm$ are now 4×4 matrices (6.3) and the unitary matrix U is

$$U = \begin{pmatrix} \frac{-iN^1_+(p_1+ip_2)}{p_4+ip_5} & \frac{-iN^1_+(p-p_3)}{p_4+ip_5} & 0 & N^1_+ \\ -iN^2_+(p_4-ip_5) & 0 & N^2_+(p-p_3) & -N^2_+(p_1-ip_2) \\ \frac{-iN^1_-(p_1+ip_2)}{p_4+ip_5} & \frac{iN^1_-(p+p_3)}{p_4+ip_5} & 0 & N^1_- \\ iN^2_-(p_4-ip_5) & 0 & N^2_-(p+p_3) & N^2_-(p_1-ip_2) \end{pmatrix}.$$

It is constructed in terms of the 2-fold degenerate eigenstates $|\psi^{(\alpha)}_\pm(p)\rangle$, $\alpha = 1,2$, of the Weyl equation

$$\mathcal{H}^5_{\mathrm{W}}|\psi^{(\alpha)}_\pm\rangle = \pm p|\psi^{(\alpha)}_\pm\rangle,$$

where the normalization constants are:

$$N^{(1)}_+ = \sqrt{\left(\frac{p_4^2+p_5^2}{2p(p-p_3)}\right)}, \quad N^{(2)}_+ = \frac{1}{\sqrt{(2p(p-p_3))}},$$
$$N^{(1)}_- = \sqrt{\left(\frac{p_4^2+p_5^2}{2p(p+p_3)}\right)}, \quad N^{(2)}_- = \frac{1}{\sqrt{(2p(p+p_3))}}.$$

One inverts the diagonalization process (6.2) and express (6.22) in terms of the projection operators (6.8) as

$$\mathcal{H}^5_{\rm W} = pU^\dagger(\mathcal{I}^+ - \mathcal{I}^-)U = p(P^+ - P^-).$$

The 4×4 matrix P^+ is explicitly calculated to be

$$P^+ = \frac{1}{2p}\begin{pmatrix} p+p_3 & p_1 - ip_2 & i(p_4+ip_5) & 0 \\ p_1+ip_2 & p-p_3 & 0 & i(p_4+ip_5) \\ -i(p_4-ip_5) & 0 & p-p_3 & -(p_1-ip_2) \\ 0 & -i(p_4-ip_5) & -(p_1+ip_2) & p+p_3 \end{pmatrix}.$$

Using (6.7) and (6.8) it is possible to redefine P^+ in terms of the $5+1$ dimensional Weyl Hamiltonian (6.22):

$$P^+ = \frac{1}{2}(\frac{\mathcal{H}^5_{\rm W}}{p} + 1). \tag{6.23}$$

The winding number in $5+1$ dimensions is defined as

$$\begin{aligned}\mathcal{C}_5 &= -\frac{i\epsilon^{\mu\nu\rho\sigma\lambda\gamma}}{48\pi^3}\int d^5p\, dw\, \mathrm{Tr}\,[(G\partial_\mu G^{-1})(G\partial_\nu G^{-1})(G\partial_\rho G^{-1})(G\partial_\sigma G^{-1})(G\partial_\lambda G^{-1})(G\partial_\gamma G^{-1})] \\ &= -\frac{i}{8\pi^3}\int d^5p\, dw\, \epsilon^{ijklm}\mathrm{Tr}\,[(G^2\partial_i G^{-1})(G\partial_j G^{-1})(G\partial_k G^{-1})(G\partial_l G^{-1})(G\partial_m G^{-1})], \end{aligned} \tag{6.24}$$

where $\mu,\nu... = 0,...,5$ and $i,j... = 1,5$. One performs the w integration and obtains

$$\mathcal{C}_5 = \frac{1}{8\pi^2}\int d^5p\, \epsilon^{ijklm}\mathrm{Tr}\,[\partial_i P^+\partial_j P^+\partial_k P^+\partial_l P^+\partial_m P^+].$$

This can be written as the total derivative,

$$\mathcal{C}_5 = \frac{1}{8\pi^2}\int d^5p\, \boldsymbol{\nabla}\cdot\boldsymbol{K}_5, \quad K^i_5 = \epsilon^{ijklm}\mathrm{Tr}\,[P^+\partial_j P^+\partial_k P^+\partial_l P^+\partial_m P^+].$$

It is possible to use (6.23) and convert K^i_5 into the simpler form:

$$\begin{aligned} K^i_5 &= \frac{1}{(2p)^5}\epsilon^{ijklm}\mathrm{Tr}\,[\mathcal{H}^5_{\rm W}(\partial_j\mathcal{H}^5_{\rm W})(\partial_k\mathcal{H}^5_{\rm W})(\partial_l\mathcal{H}^5_{\rm W})(\partial_m\mathcal{H}^5_{\rm W})] \\ &= \frac{1}{(2p)^5}\epsilon^{ijklm}\mathrm{Tr}\,[\boldsymbol{\Sigma}\cdot\boldsymbol{p}\Sigma_j\Sigma_k\Sigma_l\Sigma_m] \\ &= \frac{p^i}{(2p)^5}4!\mathrm{Tr}\,[1_{4\times 4}] = 6\frac{p^i}{2p^5}. \end{aligned}$$

One observes that the winding number (6.24), similar to the $3+1$ dimensional case, yields the Dirac monopole $\boldsymbol{b}_5 = \frac{\boldsymbol{p}}{2p^5}, \boldsymbol{\nabla}\cdot\boldsymbol{b}_5 = \frac{4\pi^2}{3}\delta^5(p)$. Therefore, it is concluded that the value of (6.24) is equal to the unit charge of the monopole:

$$\mathcal{C}_5 = \frac{3}{4\pi^2}\int d^5p\, \boldsymbol{\nabla}\cdot\boldsymbol{b}_5 = 1.$$

In [68], the same monopole field was found in terms of the 2×2 matrix Berry gauge field,

$$\mathcal{A}_i^{\alpha\beta} = i\langle\psi_+^{(\alpha)}|\frac{\partial}{\partial p^i}|\psi_+^{(\beta)}\rangle = i(\mathcal{I}^+ U\frac{\partial}{\partial p^i}U^\dagger\mathcal{I}^+)^{\alpha\beta}, \tag{6.25}$$

and its curvature

$$\mathcal{G}_{ij}^{\alpha\beta} = \partial_i\mathcal{A}_j^{\alpha\beta} - \partial_j\mathcal{A}_i^{\alpha\beta} - i[\mathcal{A}_i, \mathcal{A}_j]^{\alpha\beta},$$

as

$$\frac{1}{24}\epsilon^{ijklm}\mathrm{Tr}\,[\mathcal{G}_{jk}\mathcal{G}_{lm}] = -\frac{pi}{2p^5}. \tag{6.26}$$

Like in the $3+1$ dimensional case, this monopole field is the source of the chiral magnetic effect [68] and the chiral anomaly [34, 68] in the semiclassical chiral kinetic theory. One can accomplish the relation between the Berry gauge field (6.25) and the winding number (6.24) by substituting the explicit form of the projection operators (6.7) into K_5^i:

$$\begin{aligned} K_5^i = \epsilon^{ijklm}\mathrm{Tr} \quad & [\quad \mathcal{I}^+\partial_j U\partial_k U^\dagger\mathcal{I}^+\partial_l U\partial_m U^\dagger\mathcal{I}^+ \\ & + \quad 2\mathcal{I}^+ U\partial_j U^\dagger\mathcal{I}^+ U\partial_k U^\dagger\mathcal{I}^+\partial_l U\partial_m U^\dagger\mathcal{I}^+ \\ & + \quad \mathcal{I}^+ U\partial_j U^\dagger\mathcal{I}^+ U\partial_k U^\dagger\mathcal{I}^+ U\partial_l U^\dagger\mathcal{I}^+ U\partial_m U^\dagger\mathcal{I}^+] \\ & = \quad -\frac{1}{4}\epsilon^{ijklm}\mathrm{Tr}\,[\mathcal{G}_{jk}\mathcal{G}_{lm}]. \end{aligned}$$

Hence the winding number (6.24) and the monopole field (6.26) which is responsible for the chiral anomaly in kinetic theory are related:

$$\mathcal{C}_5 = -\frac{1}{32\pi^2}\int d^5p\;\partial_i\epsilon^{ijklm}\mathrm{Tr}\,[\mathcal{G}_{jk}\mathcal{G}_{lm}]. \tag{6.27}$$

On the other hand letting the domain of the integral (6.27) to be the 5-ball B^5 whose boundary is the 4-sphere S^4, $\mathcal{C}_5$ can be written as

$$\mathcal{C}_5 = -\frac{1}{32\pi^2}\int_{B^5} d^5p\;\epsilon^{ijklm}\partial_i(\mathrm{Tr}\,[\mathcal{G}_{jk}\mathcal{G}_{lm}]) = -\frac{1}{32\pi^2}\int_{S^4} d^4p\;\epsilon^{\mathrm{jklm}}\mathrm{Tr}\,[\mathcal{G}_{\mathrm{jk}}\mathcal{G}_{\mathrm{lm}}],$$

where $\mathrm{j,k}...$ represent the angular coordinates on S^4. Therefore it is found that the winding number is the negative of another topological invariant, namely second Chern number which is the integral of the second Chern character over S^4.

As the Dirac monopole is of concern, one would like to explore its gauge field structure. In differential form language writing the 2-form Berry field strength as

$\mathcal{G} = \frac{1}{2}\mathcal{G}_{ij}dp^i \wedge dp^j$ and recalling (6.26) the Abelian 3-form antisymmetric gauge field $\mathcal{B}_5$ [69] is defined as

$$\mathrm{Tr}\,[\mathcal{G}\mathcal{G}] = d\mathcal{B}_5.$$

$\mathcal{B}_5$ can be written explicitly as

$$\mathcal{B}_5 = \mathrm{Tr}\,[\mathcal{A}d\mathcal{A} - \frac{2i}{3}\mathcal{A}^3], \tag{6.28}$$

where $\mathcal{A}$ is the Berry gauge field (6.25). Note that (6.28) is in the form of Chern Simons Lagrangian. In its components Abelian rank-3 antisymmetric gauge field (6.28) can be written as,

$$\mathcal{B}_5^{ijk} = \mathrm{Tr}\,[\mathcal{A}^i\partial^j\mathcal{A}^k - \frac{2i}{3}\mathcal{A}^i\mathcal{A}^j\mathcal{A}^k].$$

6.5 $d+1$ Dimensional Weyl Hamiltonian and the Winding Number

In the $d+1$ dimensional spacetime where $d+1$ is even, each eigenvalue $(p,-p)$ of the Weyl Hamiltonian (6.1) is $\frac{d-1}{2}$ fold degenerate and in principle the corresponding eigenstates

$$|\psi_+^{(1)}\rangle, ..., |\psi_+^{(\frac{d-1}{2})}\rangle, \quad |\psi_-^{(1)}\rangle, ..., |\psi_-^{(\frac{d-1}{2})}\rangle,$$

can be found. One can diagonalize (6.1) in terms of the unitary matrix U which is constructed in terms of the solutions $|\psi_\pm^\alpha\rangle$ as

$$U = \big(|\psi_+^{(1)}\rangle ... |\psi_-^{(\frac{d-1}{2})}\rangle\big)^\dagger.$$

With an appropriate choice of the normalization constant N_d, it is possible to write (6.6) as

$$\mathcal{C}_d = \frac{i^{\frac{d+1}{2}} 2^{\frac{3d-5}{2}} d}{\pi(d+1)!\binom{d-1}{\frac{d+1}{2}}\mathrm{Vol}(S^{d-1})} \int d^dp\, dw\, \epsilon^{\mu\nu..\rho}\mathrm{Tr}\,[(G\partial_{p_\mu}G^{-1})(G\partial_{p_\nu}G^{-1})...(G\partial_{p_\rho}G^{-1})].$$

Actually, $\mathcal{C}_d$ should be multiplied with -1 for the $3+1$ dimensional case in order to cancel the minus factor which will appear in the K_3^M (6.31). Because of this minus sign, in $3+1$ dimensions, the calculated Chern number is equal to -1 which is different from the general construction (6.36). This sign confusion is artificial in the sense that it is due to the choice of $\mathcal{H}_{\mathrm{W}}^3 = \boldsymbol{\sigma}\cdot\boldsymbol{p}$ which is the conventional Weyl Hamiltonian used in the literature. If the Dirac matrices were constructed starting from the $1+1$ dimensions in chiral basis, there would not be this sign ambiguity. It should be emphasized that the general definition of the winding number do not distinguish between the dimensions.

Using the properties of the projection operators (6.8) and the definitions (6.9), (6.10) one can perform the w integration and obtain,

$$\begin{aligned}
\mathcal{C}_d &= \frac{i^{\frac{d+1}{2}} 2^{\frac{3d-5}{2}}}{\pi(d-1)!\binom{d-1}{\frac{d+1}{2}}\mathrm{Vol}(S^{d-1})} \int d^d p \, dw \, \epsilon^{\mathrm{MN...R}} \mathrm{Tr}\,[G^2(\partial_{\mathrm{M}} G^{-1})(G\partial_{\mathrm{N}} G^{-1})...(G\partial_{\mathrm{R}} G^{-1})] \\
&= \frac{-2(-2i)^{\frac{d-1}{2}}}{(d-1)!\mathrm{Vol}(S^{d-1})} \int d^d p \, \epsilon^{\mathrm{M...R}} \mathrm{Tr}\,[(\partial_{\mathrm{M}} P^+)...(\partial_{\mathrm{R}} P^+)] \qquad \textbf{(6.29)} \\
&= \frac{-2(-2i)^{\frac{d-1}{2}}}{(d-1)!\mathrm{Vol}(S^{d-1})} \int d^d p \, \boldsymbol{\nabla} \cdot \boldsymbol{K}_d
\end{aligned}$$

where the italic capital letters denote $M, N, .., R = 1, .., d$ and $\epsilon^{\mathrm{M...R}}$ is the d dimensional totally antisymmetric tensor. Using the properties of the projection operators (6.8) and the definitions (6.9), (6.10) one can perform the w integration and obtain,

$$\begin{aligned}
\mathcal{C}_d &= \frac{i^{\frac{d+1}{2}} 2^{\frac{3d-5}{2}}}{\pi(d-1)!\binom{d-1}{\frac{d+1}{2}}\mathrm{Vol}(S^{d-1})} \int d^d p \, dw \, \epsilon^{\mathrm{MN...R}} \mathrm{Tr}\,[G^2(\partial_{\mathrm{M}} G^{-1})(G\partial_{\mathrm{N}} G^{-1})...(G\partial_{\mathrm{R}} G^{-1})] \\
&= \frac{-2(-2i)^{\frac{d-1}{2}}}{(d-1)!\mathrm{Vol}(S^{d-1})} \int d^d p \, \epsilon^{\mathrm{M...R}} \mathrm{Tr}\,[(\partial_{\mathrm{M}} P^+)...(\partial_{\mathrm{R}} P^+)] \qquad \textbf{(6.30)} \\
&= \frac{-2(-2i)^{\frac{d-1}{2}}}{(d-1)!\mathrm{Vol}(S^{d-1})} \int d^d p \, \boldsymbol{\nabla} \cdot \boldsymbol{K}_d
\end{aligned}$$

where the italic capital letters denote $M, N, .., R = 1, .., d$ and $\epsilon^{\mathrm{M...R}}$ is the d dimensional totally antisymmetric tensor. $\boldsymbol{K}_d$ is defined in terms of the projection operators as

$$K_d^{\mathrm{M}} = \epsilon^{\mathrm{MN...R}} \mathrm{Tr}\,[P^+(\partial_{\mathrm{N}} P^+)...(\partial_{\mathrm{R}} P^+)].$$

It leads to the Dirac monopole in the d dimensional momentum space:

$$\begin{aligned}
K_d^{\mathrm{M}} &= \frac{1}{(2p)^d} \epsilon^{\mathrm{MN...R}} \mathrm{Tr}\,[\mathcal{H}_{\mathrm{W}}(\partial_{\mathrm{N}} \mathcal{H}_{\mathrm{W}})...(\partial_{\mathrm{R}} \mathcal{H}_{\mathrm{W}})] \\
&= \frac{1}{(2p)^d} \epsilon^{\mathrm{MN...R}} \mathrm{Tr}\,[\boldsymbol{\Sigma} \cdot \boldsymbol{p} \Sigma_{\mathrm{N}} ... \Sigma_{\mathrm{R}}] \qquad \textbf{(6.31)} \\
&= -(d-1)!(\frac{i}{2})^{\frac{d-1}{2}} \frac{p^{\mathrm{M}}}{2p^d}.
\end{aligned}$$

Therefore for all spacetime dimensions considered, the winding number (6.6) is associated to the Dirac monopole,

$$\boldsymbol{b}_d = \frac{\boldsymbol{p}}{2p^d}, \ \boldsymbol{\nabla} \cdot \boldsymbol{b}_d = \frac{\mathrm{Vol}(S^{d-1})}{2} \delta^d(p). \qquad \textbf{(6.32)}$$

Using the definition of the d dimensional monopole field (6.32), one can compute (6.30) as

$$\mathcal{C}_d = \frac{2}{\mathrm{Vol}(S^{d-1})} \int_{B^d} d^d p \; \boldsymbol{\nabla} \cdot \boldsymbol{b}_d = 1. \tag{6.33}$$

Thus the winding number $\mathcal{C}_d$ is equal to the unit charge of the d dimensional Dirac monopole.

To show the general relation between the winding number and the Chern number, it can be pointed out that the winding number can be written as an integral over the d dimensional ball B^d as

$$\begin{aligned}
\mathcal{C}_d &= \frac{-2(-1)^{\frac{d-1}{2}}}{\mathrm{Vol}(S^{d-1})(d-1)!} \int_{B^d} d^d p \; \epsilon^{\mathrm{MNR...ST}} \partial_{\mathrm{M}} \mathrm{Tr}\, [\overbrace{\mathcal{G}_{\mathrm{NR}} ... \mathcal{G}_{\mathrm{ST}}}^{\text{(d-1)/2 times}}] \\
&= \frac{2(-1)^{\frac{d+1}{2}}}{\mathrm{Vol}(S^{d-1})(d-1)!} \int_{S^{d-1}} d^{d-1} p \; \epsilon^{\mathrm{NR...ST}} \mathrm{Tr}\, [\overbrace{\mathcal{G}_{\mathrm{NR}} ... \mathcal{G}_{\mathrm{ST}}}^{\text{(d-1)/2 times}}],
\end{aligned}$$

where $\mathcal{G}_{\mathrm{NR}}$ is the Berry field strength (6.5) and the letters $\mathrm{M}, \mathrm{N}...$ represents the angular coordinates of the $d-1$ dimensional sphere S^{d-1}. It is found that the monopole field (6.32) which is obtained from the winding number can be expressed by means of the Berry curvature as

$$b_d^{\mathrm{M}} = \frac{(-1)^{\frac{d+1}{2}}}{(d-1)!} \epsilon^{\mathrm{MNR...ST}} \mathrm{Tr}\, [\overbrace{\mathcal{G}_{\mathrm{NR}} ... \mathcal{G}_{\mathrm{ST}}}^{\text{(d-1)/2 times}}]. \tag{6.34}$$

(6.34) is in accord with the previous results and it demonstrates that the monopole which is responsible for the chiral anomaly in chiral kinetic theory is the same with the one appearing in the winding number (6.32). Since one can define the winding number (6.6) for even $d+1$ dimensions, the existence of semiclassical chiral anomaly has been proven for all even dimensions. Let $d = 2n+1$ where n is a positive integer. The volume of the $d-1$ dimensional sphere is given in terms of Γ function as $\frac{2\pi^{d/2}}{\Gamma(d/2)}$. Using this, one finds

$$\begin{aligned}
\mathcal{C}_{2n+1} &= \frac{\Gamma(n+\frac{1}{2})(-1)^{n+1}}{\pi^{n+\frac{1}{2}}\Gamma(2n+1)} \int_{S^{2n}} d^{2n} p \; \epsilon^{\mathrm{NR...ST}} \mathrm{Tr}\, [\overbrace{\mathcal{G}_{\mathrm{NR}} ... \mathcal{G}_{\mathrm{ST}}}^{n \; times}] \\
&= \frac{(-1)^{n+1}}{(4\pi)^n n!} \int_{S^{2n}} d^{2n} p \; \epsilon^{\mathrm{NR...ST}} \mathrm{Tr}\, [\overbrace{\mathcal{G}_{\mathrm{NR}} ... \mathcal{G}_{\mathrm{ST}}}^{n \; times}] = (-1)^{n+1} \mathcal{N}_n.
\end{aligned} \tag{6.35}$$

$\mathcal{N}_n$ is the n^{th} Chern number defined as the integral of the n^{th} Chern character. Comparing it with (6.33) one concludes that,

$$\mathcal{N}_n = (-1)^{n+1}, \tag{6.36}$$

for all n. Up to a normalization, (6.36) is in accord with the spin Chern number computed in [34]. (6.35) shows that the winding number $\mathcal{C}_{2n+1}$ which is equal to the charge of the Dirac monopole, is actually the topological index given by the integral of the Berry curvature (6.5).

It is also possible to investigate the gauge field structure of the d dimensional Dirac monopole (6.32). A generalization of the Dirac magnetic monopoles to all dimensions by means of the antisymmetric tensor gauge fields was considered in [69]. Utilizing the differential forms, one defines the Abelian antisymmetric tensor gauge field $\mathcal{B}_{2n+1}$ as

$$\mathrm{Tr}\,[\mathcal{G}^n] = d\mathcal{B}_{2n+1} = d\mathrm{Tr}\,[L^{2n-1}_{CS}],$$

where L^{2n-1}_{CS} is the $2n-1$ dimensional Chern-Simons Lagrangian. In its components $\mathcal{B}_{2n+1}$ can be written as

$$\mathcal{B}_{2n+1} = \mathcal{B}^{\mathrm{M...R}}_{2n+1}\overbrace{dp_{\mathrm{M}}...dp_{\mathrm{R}}}^{2n-1\ times} = \epsilon^{\mathrm{MNR...ST}}\mathrm{Tr}\,[\mathcal{A}_{\mathrm{M}}\overbrace{\mathcal{G}_{\mathrm{NR}}...\mathcal{G}_{\mathrm{ST}}}^{n-1\ times}]\, d^{2n-1}p \qquad \textbf{(6.37)}$$

which is in accord with (6.21) and (6.28) where $\mathcal{A}_{\mathrm{M}}$ is the Berry gauge field (6.4). It should be emphasized that as (6.37) is the field of the Dirac monopole, it is not defined globally on S^{2n}.

6.6 Discussions

Diagonalizing the $2n+2$ dimensional Weyl Hamiltonian (6.1), the Berry gauge field whose gauge group is $U(n)$ is defined. The related holonomy is given as the integral of the Chern character over the compact space S^{2n}. Both in $3+1$ dimensions and in $5+1$ dimensions the fermionic winding numbers are calculated explicitly and proved that they are equal to the unit monopole charge which emerges in momentum space. It was shown that the winding number also can be stated as the integral of the Chern character, the topological invariant Chern number. The Chern character which is expressed via the Berry field strength (6.5) represents the nontrivial topological properties of the fiber bundle $(S^{2n}, U(n))$. It is observed observed that the winding numbers, monopole charges and Chern numbers are based on the same topological origin. The gauge field structure of the Dirac monopole is classified as $2n-1$ rank antisymmetric tensor gauge field as in [69].

In quantum field theory the chiral anomaly is the contribution of different chirality zero-modes to the measure of the path integral under the chiral transformation. The anomaly term is precisely equal to the analytical index of the Dirac operator which is projected to the positive chirality subspace. Due to the Atiyah-Singer index theorem, the analytical index is also topological in the sense that it can be given as the integral of the Chern class over the compact manifolds. On the other hand, in [31, 32, 34, 68] it was argued that chiral anomaly can be achieved at the semiclassical level via the chiral kinetic theory by introducing the Berry gauge field. In the context of the one particle Weyl Hamiltonians, in all even dimensions it is shown that the existence of the Dirac monopoles on the degeneracy points, that is $p = 0$. This monopole field acts as a source for the semiclassical anomaly. It is observed that the topological invariant winding number (6.6) is equal to the charge of the monopole and according to [39] this winding number results in the chirality of the Weyl particle. It is interesting but may be not surprising that even at the semiclassical level the chirality which shows up itself as a monopole located at the zero momentum degeneracy point is responsible for the chiral anomaly. Besides, this topological structure is also determined by the integral of the Chern character of the Berry curvature which is the topological index. A topological and geometrical analysis of the semiclassical chiral anomaly is done and concluded that the concepts like the topological charges, fiber bundle theory, index theorems etc. also play an important role in the semiclassical theory.

The $3 + 1$ dimensional Weyl Hamiltonian (6.11) was argued to be the effective low energy Hamiltonian for the Weyl semimetals. In [40] it was noted that the stability of the Weyl semimetallic phase is related to the Chern number which is calculated in terms of the Berry gauge field (6.20). In this work, it is shown that this stability can be determined in terms of the another topological invariant, the winding number (6.14).

7. CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusions

In Chapter 2, the free, massive Dirac Hamiltonian is diagonalized by the Foldy-Wouthuysen transformation. A pure gauge field is extracted though the diagonalization. By projecting onto the positive energy subspace, the Berry gauge field is obtained. In $2+1$ and $4+1$ dimensions Berry field strengths are acquired and the related Chern numbers are calculated. In order to construct the effective field theory of the Kane-Mele model of the $2+1$ dimensional topological insulator, the fermionic degrees of freedom are integrated out through the path integral formalism. The resultant theory is the $2+1$ dimensional Chern-Simons action of the external electromagnetic fields. The coefficient of the effective action is topological in the sense that it is equal to the winding number of the free fermion propagator. The conductivity of the Kane-Mele model of the $2+1$ dimensional topological insulator is related to the coefficient of the resulting effective theory. It is shown that both in $2+1$ dimensions and $4+1$ dimensions the winding numbers are equal to the related Chern numbers of the Berry curvatures.

In Chapter 3, the effective field theory of the $2+1$ dimensional spin Hall insulator in the presence of the Rashba spin-orbit interaction is constructed. Kane-Mele Hamiltonian of the spin Hall insulator is diagonalized and the Berry gauge field is calculated. Dealing with the massive Dirac particle coupled to the external spin and electromagnetic gauge fields, the fermions are integrated via the path integral quantization. Although the spin is not conserved whenever the Rashba interaction is switched on, one can still deal with the spin Hall conductivity. It is demonstrated that the effective field theory is of BF type. In order to obtain the spin current, the the effective theory is varied with the spin gauge field. It is shown that the the spin Hall conductivity is related to the winding number of the fermion propagator. It is shown numerically that the effect of the Rashba interaction slightly changes the quantized spin Hall conductivity.

In Chapter 4, a model composed of neutral Dirac fermions is proposed for the $3+1$ dimensional topological insulators. After a regularization procedure, the θ term for the axion electrodynamics is obtained as the effective action. $3+1$ topological insulators are also discussed in the context of BF type theories.

In Chapter 5, the semiclassical chiral kinetic theory is considered. The massless Dirac Hamiltonian is diagonalized and the Berry gauge field is calculated explicitly in $3+1$ and $5+1$ dimensions. In all even $d+1$ spacetime dimensions it is demonstrated that the related Berry curvatures yield the field of a Dirac monopole that is located at the very center of the momentum space. The semiclassical chiral anomaly is attained through the Liouville equation. By means of the semiclassical equations of motion for the phase space variables, the chiral current is constructed and CME is acquired successfully. It is demonstrated that the Dirac monopole is responsible for both the semiclassical chiral anomaly and the CME. The existence of the chiral anomaly and the CME is formulated within the same theory for all even $d+1$ dimensions. Moreover an efficient method of finding the modified phase space measure is presented.

In Chapter 6, the related Berry gauge fields are calculated through the diagonalization of the $3+1$ and $5+1$ dimensional Weyl Hamiltonians. The Berry curvatures are calculated explicitly. Both in $3+1$ and $5+1$ dimensions, the fermionic winding numbers are calculated and it is shown that they are equal to the unit monopole charge in momentum space. It is proven that this monopole is the same with the Dirac monopole that results from the Berry curvature. The Dirac monopole charge is also equal to the Chern number. Hence the monopole charge has a topological origin. All the arguments are generalized to even $d+1$ dimensions. It is demonstrated explicitly that both the semiclassical chiral anomaly and the CME stem from this topological charge which represents the chirality of the Hamiltonian. Moreover, this topological structure is also determined by the integral of the Chern character of the Berry curvature which is the topological index. It is concluded that the concepts like the topological charges, fiber bundle theory, index theorems etc. also play an important role in the semiclassical theory.

7.2 Recommendations and Future Directions

The effective field theory approach to the monolayer and bilayer graphene may need more clarification. One may find interesting to construct the field theories representing the effects of the substrate, gravity, strain as gauge fields.

The $3 + 1$ dimensional Weyl Hamiltonian was argued to be the effective low energy Hamiltonian for the Weyl semimetals. It is supposed that the stability of the Weyl semimetallic phase is related to the Chern number which is calculated in terms of the Berry gauge field. It would be interesting to discuss the roles of these topological numbers in the effective field theory approach to the Weyl semimetals.

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APPENDICES

APPENDIX A.1 : Explicit calculation of the coefficient C_s
APPENDIX A.2 : The Berry curvature in $5+1$ dimensions
APPENDIX A.3 : Properties of the Berry Curvature and the Σ Matrices in $d+1$ dimensions

APPENDIX A.1 Explicit calculation of the coefficient

First of all one can observe that (3.27) can be expressed in terms of $P^{MN} \equiv P^M S_z P^N + P^N S_z P^M$, so that it can be amalgamated with (3.28) to write C_s as

$$
\begin{aligned}
C_s \;=\; & -\frac{i}{48\pi^2}\int d^2p\mathrm{tr}\Big\{P^{57}\left[(1+\frac{E_1-E_7}{E_1-E_5}+\frac{E_1-E_5}{E_1-E_7})\mathcal{I}^1+(1+\frac{E_3-E_7}{E_3-E_5}+\frac{E_3-E_5}{E_3-E_7})\mathcal{I}^3\right]\\
&+P^{68}\left[(1+\frac{E_2-E_8}{E_2-E_6}+\frac{E_2-E_6}{E_2-E_8})\mathcal{I}^2+(1+\frac{E_4-E_8}{E_4-E_6}+\frac{E_4-E_6}{E_4-E_8})\mathcal{I}^4\right]\\
&-P^{13}\left[(1+\frac{E_5-E_3}{E_5-E_1}+\frac{E_5-E_1}{E_5-E_3})\mathcal{I}^5+(1+\frac{E_7-E_3}{E_7-E_1}+\frac{E_7-E_1}{E_7-E_3})\mathcal{I}^7\right]\\
&-P^{24}\left[(1+\frac{E_6-E_4}{E_6-E_2}+\frac{E_6-E_2}{E_6-E_4})\mathcal{I}^6+(1+\frac{E_8-E_4}{E_8-E_2}+\frac{E_8-E_2}{E_8-E_4})\mathcal{I}^8\right]\\
&+P^{35}\left[(\frac{E_1-E_3}{E_5-E_3}+\frac{E_1-E_3}{E_1-E_5})\mathcal{I}^1-(\frac{E_7-E_5}{E_3-E_5}+\frac{E_7-E_5}{E_7-E_3})\mathcal{I}^7\right]\\
&+P^{46}\left[(\frac{E_2-E_4}{E_6-E_4}+\frac{E_2-E_4}{E_2-E_6})\mathcal{I}^2-(\frac{E_8-E_6}{E_4-E_6}+\frac{E_8-E_6}{E_8-E_4})\mathcal{I}^8\right]\\
&+P^{17}\left[(\frac{E_3-E_1}{E_7-E_1}+\frac{E_3-E_1}{E_3-E_7})\mathcal{I}^3-(\frac{E_5-E_7}{E_1-E_7}+\frac{E_5-E_7}{E_5-E_1})\mathcal{I}^5\right]\\
&+P^{28}\left[(\frac{E_4-E_2}{E_8-E_2}+\frac{E_4-E_2}{E_4-E_8})\mathcal{I}^4-(\frac{E_6-E_8}{E_2-E_8}+\frac{E_6-E_8}{E_6-E_2})\mathcal{I}^6\right]\Big\}.
\end{aligned}
$$

Making use of the polar coordinates (3.25) and the definitions (3.14) one can show that it can be written in the form

$$
\begin{aligned}
C_s \;=\; & -\frac{1}{12\pi^2}\int d^2p\Big[\frac{F_5^2F_7^2pA(5,7)}{(\Delta_{SO}-E_1)(\Delta_{SO}-E_5)(\Delta_{SO}-E_7)} \\
& (1+\frac{E_1-E_7}{E_1-E_5}+\frac{E_1-E_5}{E_1-E_7})(E_5-E_1)\frac{\partial F_1^2}{\partial p} \\
& -\frac{F_6^2F_8^2C(6,8)}{p(\Delta_{SO}-E_2)}(1+\frac{E_2-E_8}{E_2-E_6}+\frac{E_2-E_6}{E_2-E_8})(E_6-E_2)\frac{\partial F_2^2}{\partial p} \\
& +\frac{F_5^2F_7^2pA(5,7)}{(\Delta_{SO}-E_3)(\Delta_{SO}-E_5)(\Delta_{SO}-E_7)}(1+\frac{E_3-E_7}{E_3-E_5}+\frac{E_3-E_5}{E_3-E_7})(E_7-E_3)\frac{\partial F_3^2}{\partial p} \\
& -\frac{F_6^2F_8^2C(6,8)}{p(\Delta_{SO}-E_4)}(1+\frac{E_4-E_8}{E_4-E_6}+\frac{E_4-E_6}{E_4-E_8})(E_8-E_4)\frac{\partial F_4^2}{\partial p} \\
& +\frac{F_1^2F_3^2pA(1,3)}{(\Delta_{SO}-E_1)(\Delta_{SO}-E_3)(\Delta_{SO}-E_5)}(1-\frac{E_5-E_3}{E_1-E_5}-\frac{E_5-E_1}{E_3-E_5})(E_5-E_1)\frac{\partial F_5^2}{\partial p} \\
& -\frac{F_2^2F_4^2C(2,4)}{p(\Delta_{SO}-E_6)}(1-\frac{E_6-E_4}{E_2-E_6}-\frac{E_6-E_2}{E_4-E_6})(E_6-E_2)\frac{\partial F_6^2}{\partial p} \\
& +\frac{F_1^2F_3^2pA(1,3)}{(\Delta_{SO}-E_1)(\Delta_{SO}-E_3)(\Delta_{SO}-E_7)}(1-\frac{E_7-E_3}{E_1-E_7}-\frac{E_7-E_1}{E_3-E_7})(E_7-E_3)\frac{\partial F_7^2}{\partial p} \\
& -\frac{F_2^2F_4^2C(2,4)}{p(\Delta_{SO}-E_8)}(1-\frac{E_8-E_4}{E_2-E_8}-\frac{E_8-E_2}{E_4-E_8})(E_8-E_4)\frac{\partial F_8^2}{\partial p} \\
& +\frac{F_3^2F_5^2pA(3,5)}{(\Delta_{SO}-E_1)(\Delta_{SO}-E_3)(\Delta_{SO}-E_5)}(\frac{E_1-E_3}{E_5-E_3}+\frac{E_1-E_3}{E_1-E_5})(E_5-E_1)\partial F_1^2\partial p \\
& -\frac{F_4^2F_6^2C(4,6)}{p(\Delta_{SO}-E_2)}(\frac{E_2-E_4}{E_6-E_4}+\frac{E_2-E_4}{E_2-E_6})(E_6-E_2)\frac{\partial F_2^2}{\partial p} \\
& +\frac{F_1^2F_7^2pA(1,7)}{(\Delta_{SO}-E_1)(\Delta_{SO}-E_3)(\Delta_{SO}-E_7)}(\frac{E_3-E_1}{E_7-E_1}+\frac{E_3-E_1}{E_3-E_7})(E_7-E_3)\frac{\partial F_3^2}{\partial p} \\
& -\frac{F_2^2F_8^2C(2,8)}{p(\Delta_{SO}-E_4)}(\frac{E_4-E_2}{E_8-E_2}+\frac{E_4-E_2}{E_4-E_8})(E_8-E_4)\frac{\partial F_4^2}{\partial p} \\
& +\frac{F_1^2F_7^2pA(1,7)}{(\Delta_{SO}-E_1)(\Delta_{SO}-E_5)(\Delta_{SO}-E_7)}(\frac{E_5-E_7}{E_1-E_7}-\frac{E_5-E_7}{E_1-E_5})(E_5-E_1)\frac{\partial F_5^2}{\partial p} \\
& -\frac{F_2^2F_8^2C(2,8)}{p(\Delta_{SO}-E_6)}(\frac{E_6-E_8}{E_2-E_8}-\frac{E_6-E_8}{E_2-E_6})(E_6-E_2)\frac{\partial F_6^2}{\partial p} \\
& +\frac{F_3^2F_5^2pA(3,5)}{(\Delta_{SO}-E_3)(\Delta_{SO}-E_5)(\Delta_{SO}-E_7)}(\frac{E_7-E_5}{E_3-E_5}-\frac{E_7-E_5}{E_3-E_7})(E_7-E_3)\frac{\partial F_7^2}{\partial p} \\
& -\frac{F_4^2F_6^2C(4,6)}{p(\Delta_{SO}-E_8)}(\frac{E_8-E_6}{E_4-E_6}-\frac{E_8-E_6}{E_4-E_8})(E_8-E_4)\frac{\partial F_8^2}{\partial p}\Big],
\end{aligned}
$$

where we defined

$$A(m,n)=2\left[1+\frac{p^2}{(\Delta_{SO}-E_m)(\Delta_{SO}-E_n)}\right],\; C(m,n)=2\left[1+\frac{(\Delta_{SO}-E_m)(\Delta_{SO}-E_n)}{p^2}\right].$$

Expressing $A(m,n)$ and $C(m,n)$ in terms of the normalization factors (3.14), it can further be simplified as

$$\begin{aligned}
C_s \;=\; & -\frac{1}{6\pi^2}\int d^2p\Big[\frac{(F_5F_7+F_6F_8)F_6F_8}{p(\Delta_{SO}-E_1)}\left(1+\frac{E_1-E_7}{E_1-E_5}+\frac{E_1-E_5}{E_1-E_7}\right)(E_5-E_1)\partial_p F_1^2 \\
& -\frac{(F_5F_7+F_6F_8)F_6F_8}{p(\Delta_{SO}-E_2)}\left(1+\frac{E_2-E_8}{E_2-E_6}+\frac{E_2-E_6}{E_2-E_8}\right)(E_6-E_2)\partial_p F_2^2 \\
& +\frac{(F_5F_7+F_6F_8)F_6F_8}{p(\Delta_{SO}-E_3)}\left(1+\frac{E_3-E_7}{E_3-E_5}+\frac{E_3-E_5}{E_3-E_7}\right)(E_7-E_3)\partial_p F_3^2 \\
& -\frac{(F_5F_7+F_6F_8)F_6F_8}{p(\Delta_{SO}-E_4)}\left(1+\frac{E_4-E_8}{E_4-E_6}+\frac{E_4-E_6}{E_4-E_8}\right)(E_8-E_4)\partial_p F_4^2 \\
& +\frac{(F_1F_3+F_2F_4)F_2F_4}{p(\Delta_{SO}-E_5)}\left(1-\frac{E_5-E_3}{E_1-E_5}-\frac{E_5-E_1}{E_3-E_5}\right)(E_5-E_1)\partial_p F_5^2 \\
& -\frac{(F_1F_3+F_2F_4)F_2F_4}{p(\Delta_{SO}-E_6)}\left(1-\frac{E_6-E_4}{E_2-E_6}-\frac{E_6-E_2}{E_4-E_6}\right)(E_6-E_2)\partial_p F_6^2 \\
& +\frac{(F_1F_3+F_2F_4)F_2F_4}{p(\Delta_{SO}-E_7)}\left(1-\frac{E_7-E_3}{E_1-E_7}-\frac{E_7-E_1}{E_3-E_7}\right)(E_7-E_3)\partial_p F_7^2 \\
& -\frac{(F_1F_3+F_2F_4)F_2F_4}{p(\Delta_{SO}-E_8)}\left(1-\frac{E_8-E_4}{E_2-E_8}-\frac{E_8-E_2}{E_4-E_8}\right)(E_8-E_4)\partial_p F_8^2 \\
& +\frac{(F_4F_6-F_3F_5)F_4F_6}{p(\Delta_{SO}-E_1)}\left(\frac{E_1-E_3}{E_5-E_3}+\frac{E_1-E_3}{E_1-E_5}\right)(E_5-E_1)\partial_p F_1^2 \\
& -\frac{(F_4F_6-F_3F_5)F_4F_6}{p(\Delta_{SO}-E_2)}\left(\frac{E_2-E_4}{E_6-E_4}+\frac{E_2-E_4}{E_2-E_6}\right)(E_6-E_2)\partial_p F_2^2 \\
& +\frac{(F_2F_8-F_1F_7)F_2F_8}{p(\Delta_{SO}-E_3)}\left(\frac{E_3-E_1}{E_7-E_1}+\frac{E_3-E_1}{E_3-E_7}\right)(E_7-E_3)\partial_p F_3^2 \\
& -\frac{(F_2F_8-F_1F_7)F_2F_8}{p(\Delta_{SO}-E_4)}\left(\frac{E_4-E_2}{E_8-E_2}+\frac{E_4-E_2}{E_4-E_8}\right)(E_8-E_4)\partial_p F_4^2 \\
& +\frac{(F_2F_8-F_1F_7)F_2F_8}{p(\Delta_{SO}-E_5)}\left(\frac{E_5-E_7}{E_1-E_7}-\frac{E_5-E_7}{E_1-E_5}\right)(E_5-E_1)\partial_p F_5^2 \\
& -\frac{(F_2F_8-F_1F_7)F_2F_8}{p(\Delta_{SO}-E_6)}\left(\frac{E_6-E_8}{E_2-E_8}-\frac{E_6-E_8}{E_2-E_6}\right)(E_6-E_2)\partial_p F_6^2 \\
& +\frac{(F_4F_6-F_3F_5)F_4F_6}{p(\Delta_{SO}-E_7)}\left(\frac{E_7-E_5}{E_3-E_5}-\frac{E_7-E_5}{E_3-E_7}\right)(E_7-E_3)\partial_p F_7^2 \\
& -\frac{(F_4F_6-F_3F_5)F_4F_6}{p(\Delta_{SO}-E_8)}\left(\frac{E_8-E_6}{E_4-E_6}-\frac{E_8-E_6}{E_4-E_8}\right)(E_8-E_4)\partial_p F_8^2\Big].
\end{aligned}$$

Considering the relations (3.11) and (3.15) and performing the θ integral, C_s can be written as,

$$
\begin{aligned}
C_s \;=\; & -\frac{2}{3\pi}\int dp\Big[\frac{(F_5F_7+F_6F_8)F_6F_8}{\Delta_{SO}-E_1}(1+\frac{E_1-E_7}{E_1-E_5}+\frac{E_1-E_5}{E_1-E_7})(E_5-E_1)\partial_p F_1^2 \\
& +\frac{(F_5F_7+F_6F_8)F_6F_8}{\Delta_{SO}-E_3}(1+\frac{E_3-E_7}{E_3-E_5}+\frac{E_3-E_5}{E_3-E_7})(E_7-E_3)\partial_p F_3^2 \\
& +\frac{(F_1F_3+F_2F_4)F_2F_4}{\Delta_{SO}-E_5}(1-\frac{E_5-E_3}{E_1-E_5}-\frac{E_5-E_1}{E_3-E_5})(E_5-E_1)\partial_p F_5^2 \\
& +\frac{(F_1F_3+F_2F_4)F_2F_4}{\Delta_{SO}-E_7}(1-\frac{E_7-E_3}{E_1-E_7}-\frac{E_7-E_1}{E_3-E_7})(E_7-E_3)\partial_p F_7^2 \\
& +\frac{(F_4F_6-F_3F_5)F_4F_6}{\Delta_{SO}-E_1}(\frac{E_1-E_3}{E_5-E_3}+\frac{E_1-E_3}{E_1-E_5})(E_5-E_1)\partial_p F_1^2 \\
& +\frac{(F_2F_8-F_1F_7)F_2F_8}{\Delta_{SO}-E_3}(\frac{E_3-E_1}{E_7-E_1}+\frac{E_3-E_1}{E_3-E_7})(E_7-E_3)\partial_p F_3^2 \\
& +\frac{(F_2F_8-F_1F_7)F_2F_8}{\Delta_{SO}-E_5}(\frac{E_5-E_7}{E_1-E_7}-\frac{E_5-E_7}{E_1-E_5})(E_5-E_1)\partial_p F_5^2 \\
& +\frac{(F_4F_6-F_3F_5)F_4F_6}{\Delta_{SO}-E_7}(\frac{E_7-E_5}{E_3-E_5}-\frac{E_7-E_5}{E_3-E_7})(E_7-E_3)\partial_p F_7^2\Big].
\end{aligned}
$$

Finally by making use of relations like

$$1-\frac{F_3F_5}{F_4F_6}=\frac{E_1-E_3}{E_1-\Delta_{SO}},$$

(3.29) is accomplished.

APPENDIX A.2 The Berry curvature in $5+1$ dimensions

The field strength of the Berry gauge field (5.18) is calculated as follows,

$$
\begin{aligned}
\mathcal{G}_{12} &= \tfrac{1}{2p^3(p-p_3)}\begin{pmatrix} (p_3^2+p_4^2+p_5^2)-pp_3 & -(p_1+ip_2)\sqrt{(p_4^2+p_5^2)} \\ -(p_1-ip_2)\sqrt{(p_4^2+p_5^2)} & -(p_3^2+p_4^2+p_5^2)+pp_3 \end{pmatrix}, \\
\mathcal{G}_{13} &= \tfrac{1}{2p^3}\begin{pmatrix} p_2 & i\sqrt{(p_4^2+p_5^2)} \\ -i\sqrt{(p_4^2+p_5^2)} & -p_2 \end{pmatrix}, \\
\mathcal{G}_{14} &= \tfrac{1}{2p^3(p-p_3)}\begin{pmatrix} p_1p_5-p_2p_4 & -\frac{(ip_4+p_5)\left(i(p_1p_2+p_4p_5)+p_3p-p_2^2-p_3^2-p_5^2\right)}{\sqrt{(p_4^2+p_5^2)}} \\ \frac{(-ip_4+p_5)\left(i(p_1p_2+p_4p_5)-p_3p+p_2^2+p_3^2+p_5^2\right)}{\sqrt{(p_4^2+p_5^2)}} & p_2p_4-p_1p_5 \end{pmatrix}, \\
\mathcal{G}_{15} &= \tfrac{1}{2p^3(p-p_3)}\begin{pmatrix} -p_1p_4-p_2p_5 & \frac{(p_4-ip_5)\left(i(p_1p_2-p_4p_5)+p_3p-p_2^2-p_3^2-p_4^2\right)}{\sqrt{(p_4^2+p_5^2)}} \\ \frac{(p_4+ip_5)\left(-i(p_1p_2-p_4p_5)+p_3p-p_2^2-p_3^2-p_4^2\right)}{\sqrt{(p_4^2+p_5^2)}} & p_1p_4+p_2p_5 \end{pmatrix}, \\
\mathcal{G}_{23} &= \tfrac{1}{2p^3}\begin{pmatrix} -p_1 & -\sqrt{(p_4^2+p_5^2)} \\ -\sqrt{(p_4^2+p_5^2)} & p_1 \end{pmatrix}, \\
\mathcal{G}_{24} &= \tfrac{1}{2p^3(p-p_3)}\begin{pmatrix} p_1p_4+p_2p_5 & \frac{(p_4-ip_5)\left(i(p_4p_5-p_1p_2)+p_3p-p_1^2-p_3^2-p_5^2\right)}{\sqrt{(p_4^2+p_5^2)}} \\ \frac{(p_4+ip_5)\left(-i(p_4p_5-p_1p_2)+p_3p-p_1^2-p_3^2-p_5^2\right)}{\sqrt{(p_4^2+p_5^2)}} & -p_1p_4-p_2p_5 \end{pmatrix}, \\
\mathcal{G}_{25} &= \tfrac{1}{2p^3(p-p_3)}\begin{pmatrix} p_1p_5-p_2p_4 & \frac{(ip_4+p_5)\left(-i(p_4p_5+p_1p_2)+p_3p-p_1^2-p_3^2-p_4^2\right)}{\sqrt{(p_4^2+p_5^2)}} \\ \frac{(-ip_4+p_5)\left(i(p_4p_5+p_1p_2)+p_3p-p_1^2-p_3^2-p_4^2\right)}{\sqrt{(p_4^2+p_5^2)}} & -p_1p_5+p_2p_4 \end{pmatrix}, \\
\mathcal{G}_{34} &= \tfrac{1}{2p^3}\begin{pmatrix} -p_5 & \frac{i(p_1+ip_2)\sqrt{(p_4^2+p_5^2)}}{p_4+ip_5} \\ \frac{-i(p_1-ip_2)\sqrt{(p_4^2+p_5^2)}}{p_4-ip_5} & p_5 \end{pmatrix}, \\
\mathcal{G}_{35} &= \tfrac{1}{2p^3}\begin{pmatrix} p_4 & \frac{-(p_1+ip_2)\sqrt{(p_4^2+p_5^2)}}{p_4+ip_5} \\ \frac{-(p_1-ip_2)\sqrt{(p_4^2+p_5^2)}}{p_4-ip_5} & -p_4 \end{pmatrix}, \\
\mathcal{G}_{45} &= \tfrac{1}{2p^3(p-p_3)}\begin{pmatrix} (p_1^2+p_2^2+p_3^2)-pp_3 & (p_1+ip_2)\sqrt{(p_4^2+p_5^2)} \\ (p_1-ip_2)\sqrt{(p_4^2+p_5^2)} & -(p_1^2+p_2^2+p_3^2)+pp_3 \end{pmatrix}.
\end{aligned}
$$

APPENDIX A.3 Properties of the Berry Curvature and the Σ Matrices in $d+1$ dimensions

Interrelation between the Dirac monopole and the Berry curvature was established for an even $d+1$ dimensional Weyl Hamiltonian in [65]. We review the arguments of [65] and also demonstrate the trace properties of the wedge products of the Berry curvature $\mathcal{G}$ used in Section 5.5.

In terms of the energy eigenstates one can introduce a unitary matrix U which diagonalizes the Weyl Hamiltonian (5.3),

$$U\mathcal{H}_{\mathrm{W}}U^{\dagger} = p(\mathcal{I}^{+} - \mathcal{I}^{-}).$$

$\mathcal{I}^{+}$ and $\mathcal{I}^{-}$ project onto the positive and negative energy subspaces. The Berry gauge field (5.4) can be written by means of U as

$$\mathcal{A} = i\mathcal{I}^{+}U\partial_{\boldsymbol{p}}U^{\dagger}\mathcal{I}^{+}.$$

Thus, one can construct the Berry curvature (5.5) in terms of U.

In $d = 2n+1$ dimensions the trace of the $m \leq n$ subsequent Berry field strengths can be expressed as

$$\epsilon_{\mathrm{A_1A_2...A_{2m+1}...A_{2n+1}}}\mathrm{Tr}\,[\mathcal{G}_{\mathrm{A_2A_3}}...\mathcal{G}_{\mathrm{A_{2m}A_{2m+1}}}] = (2i)^m\epsilon_{\mathrm{A_1A_2...A_{2m+1}...A_{2n+1}}}\mathrm{Tr}\,[P^{+}(\partial_{\mathrm{A_2}}P^{+})...(\partial_{\mathrm{A_{2m+1}}}P^{+})]. \quad \textbf{(A.1)}$$

We introduced

$$P^{+} = U^{\dagger}\mathcal{I}^{+}U,$$

which can be written as

$$P^{+} = \frac{1}{2}(\frac{\mathcal{H}_{\mathrm{W}}}{p} + 1).$$

Plugging it into (**A.1**) leads to

$$\begin{aligned} & \epsilon_{\mathrm{A_1A_2...A_{2m+1}...A_{2n+1}}}\mathrm{Tr}\,[P^{+}(\partial_{\mathrm{A_2}}P^{+})\ldots(\partial_{\mathrm{A_{2m+1}}}P^{+})] \\ = & \frac{\epsilon_{\mathrm{A_1A_2...A_{2m+1}...A_{2n+1}}}}{(2p)^{2m+1}}\mathrm{Tr}\,[\mathcal{H}_{\mathrm{W}}(\partial_{\mathrm{A_2}}\mathcal{H}_{\mathrm{W}})\ldots(\partial_{\mathrm{A_{2m+1}}}\mathcal{H}_{\mathrm{W}})] \\ = & \frac{\epsilon_{\mathrm{A_1A_2...A_{2m+1}...A_{2n+1}}}}{(2p)^{2m+1}}\mathrm{Tr}\,[\boldsymbol{\Sigma}\cdot\boldsymbol{p}\Sigma_{\mathrm{A_2}}\ldots\Sigma_{\mathrm{A_{2m+1}}}]. \end{aligned} \quad \textbf{(A.2)}$$

Inspecting (**A.1**) and (**A.2**) one observes that the trace of the wedge products of the Berry curvature, $\mathcal{G}$, can be expressed in terms of the trace of the antisymmetrized Σ matrices:

$$\epsilon_{\mathrm{A_1A_2...A_{2m+1}...A_{2n+1}}}\mathrm{Tr}\,[\mathcal{G}_{\mathrm{A_2A_3}}\ldots\mathcal{G}_{\mathrm{A_{2m}A_{2m+1}}}] = (2i)^m\frac{p_{\mathrm{A}}}{(2p)^{2m+1}}\epsilon_{\mathrm{A_1A_2...A_{2m+1}...A_{2n+1}}}\mathrm{Tr}\,[\Sigma_{\mathrm{A}}\Sigma_{\mathrm{A_2}}\ldots\Sigma_{\mathrm{A_{2m+1}}}].$$

Σ_{A} obey the Clifford algebra,

$$\{\Sigma_{\mathrm{A}},\Sigma_{\mathrm{B}}\} = 2\delta_{\mathrm{AB}}.$$

Moreover, they are traceless,

$$\mathrm{Tr}\,[\Sigma_{\mathrm{A}}] = 0,$$

and in $2n+2$ dimensional spacetime they satisfy the following identity,

$$\Sigma_1...\Sigma_{2n+1} = i^{n+2}\mathbf{1}_{2^n\times 2^n}.$$

Thus the trace of $2n+1$ antisymmetric product of the Σ matrices yields

$$\frac{1}{(2n+1)!}\epsilon_{\mathrm{A_1...A_{2n+1}}}\mathrm{Tr}\,[\Sigma_{\mathrm{A_1}}\ldots\Sigma_{\mathrm{A_{2n+1}}}] = i^{n+2}2^n.$$

Actually one can observe that the trace of the product of even number of different Σ matrices always vanishes because of satisfying the Clifford algebra:

$$\mathrm{Tr}\left[\Sigma_{\mathrm{A}_1} \dots \Sigma_{\mathrm{A}_{2m}}\right] = 0.$$

Moreover, it can be easily shown that the trace of the product of $2m+1$ different Σ matrices is equal to the trace of the product of the remaining $2(n-m)$ Σ matrices which is equal to zero. Therefore, the trace of the antisymmetrized product of the Berry field strength vanishes

$$\epsilon^{\mathrm{A}_1\mathrm{A}_2\dots\mathrm{A}_{2m-1}\mathrm{A}_{2m}\dots\mathrm{A}_{2n+1}}\mathrm{Tr}\left[\mathcal{G}_{\mathrm{A}_1\mathrm{A}_2}\dots\mathcal{G}_{\mathrm{A}_{2m-1}\mathrm{A}_{2m}}\right] = 0,$$

for the case $m<n$. When $m=n$ one finds

$$\epsilon_{\mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\dots\mathrm{A}_{2n}\mathrm{A}_{2n+1}}\mathrm{Tr}\left[\mathcal{G}_{\mathrm{A}_2\mathrm{A}_3}\dots\mathcal{G}_{\mathrm{A}_{2n}\mathrm{A}_{2n+1}}\right] = (-1)^{n+1}(2n)!\frac{p_{\mathrm{A}_1}}{2p^{2n+1}},$$

which is the Dirac monopole field.

PHOTO

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